Closure operators associated to partially ordered sets

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Non Classical Models of Automata and Applications

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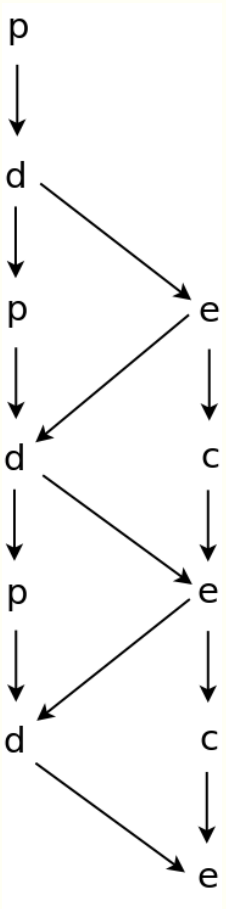
Semantics of concurrency based on partial orders

A producer-consumer system

```
P = produce -> deposit -> P
C = extract -> consume -> C
B = deposit -> extract -> B

S = (P | C | B)
```

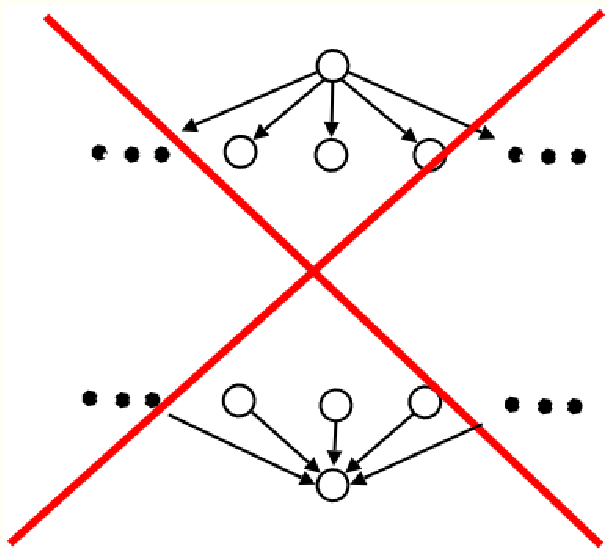


Partially Ordered Sets

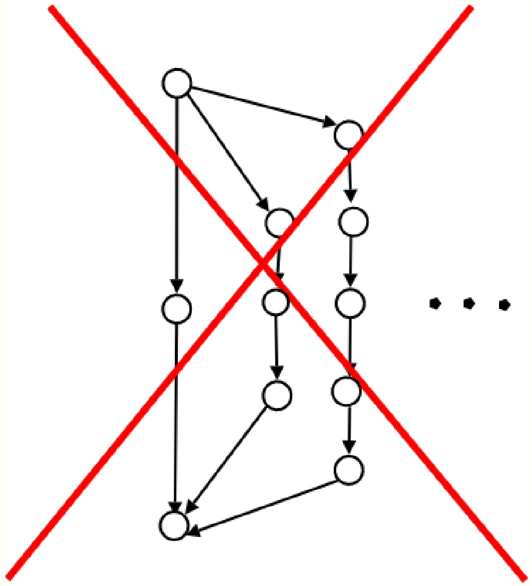
$$(X, \leq) \qquad li = \leq \cup \geq \qquad co = (X \times X) \setminus li$$

li and co are similarities (symmetric, non-transitive relations)

degree finite



interval finite

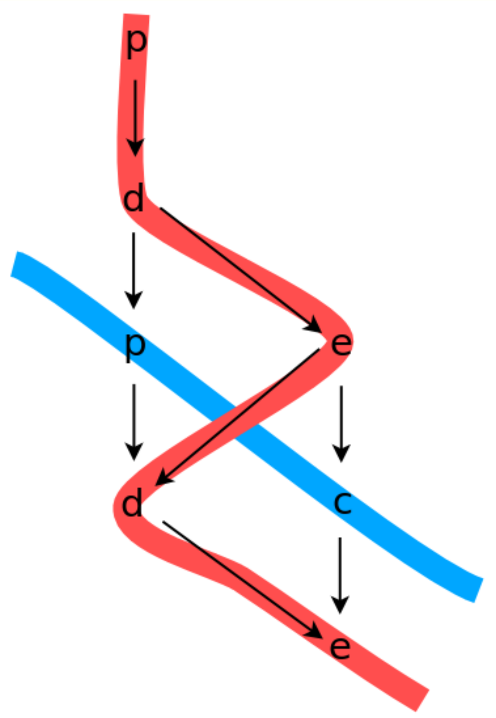


K-density

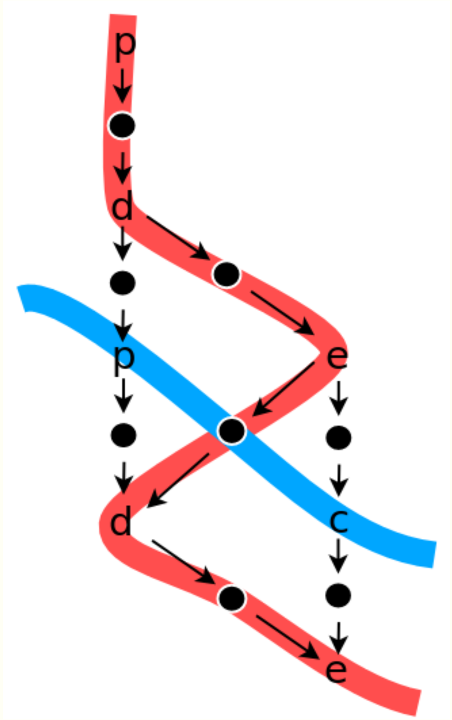
$$\forall\ line, \forall\ cut\ :\ line \cap cut \neq \emptyset$$

($line$ maximal clique of li; cut maximal clique of co)

non K-dense



K-dense



Occurrence Nets

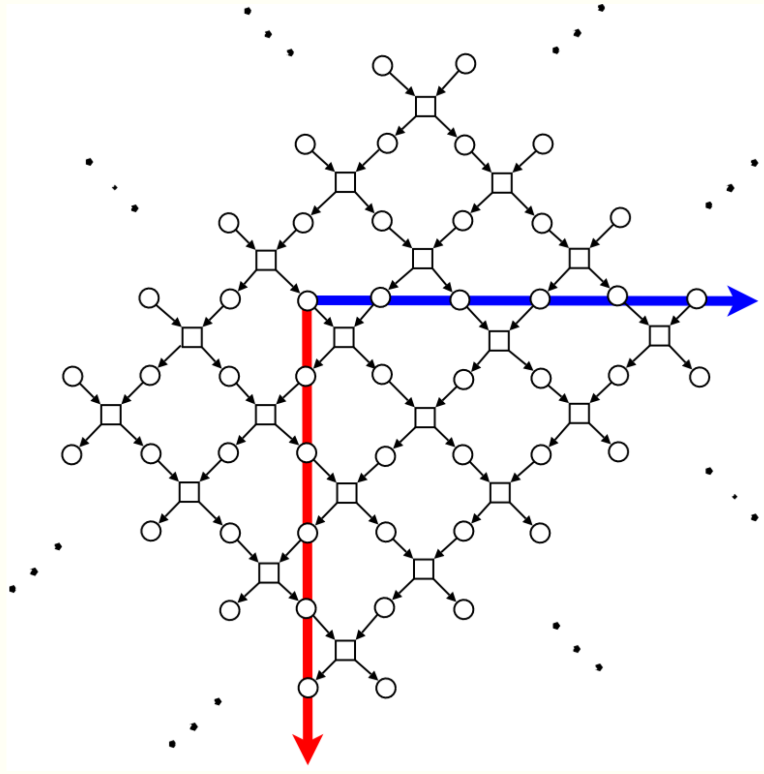
Occurrence nets are a visual representation for a space of flow of signals whose fundamental axes are causality and independence.

$$N = (B, E, F)$$

$B =$ *local states*
$E =$ *events*

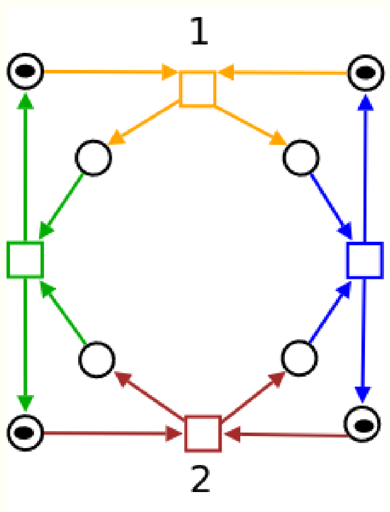
$X = B \cup E$
$\leq \, = F^*$

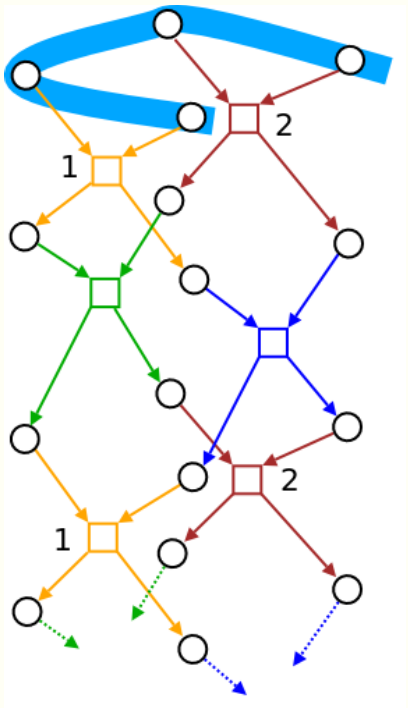
$(X, \leq), \quad \text{li}, \quad \text{co}$



Occurrence Nets as processes of Net Systems

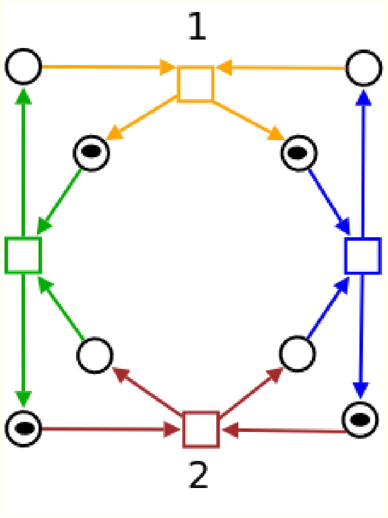
Events 1 and 2 enabled

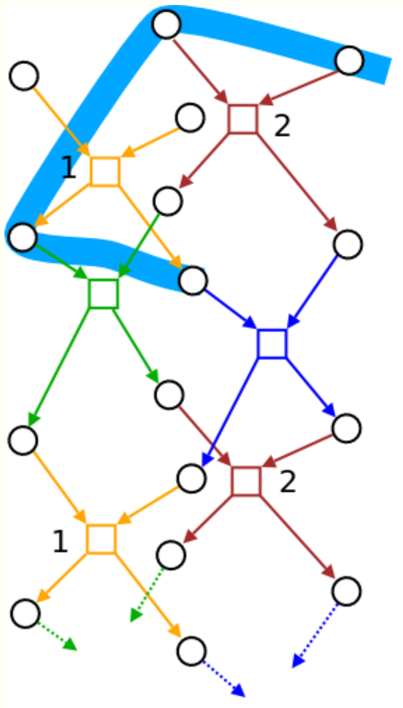




Occurrence Nets as processes of Net Systems

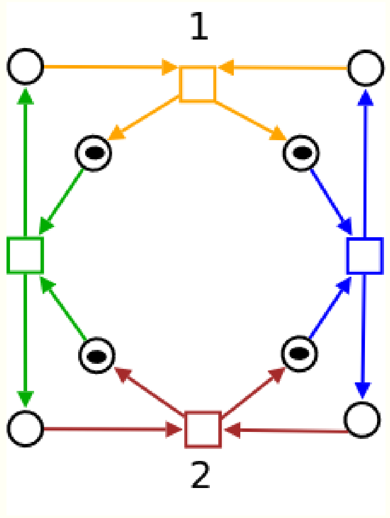
Case after 1

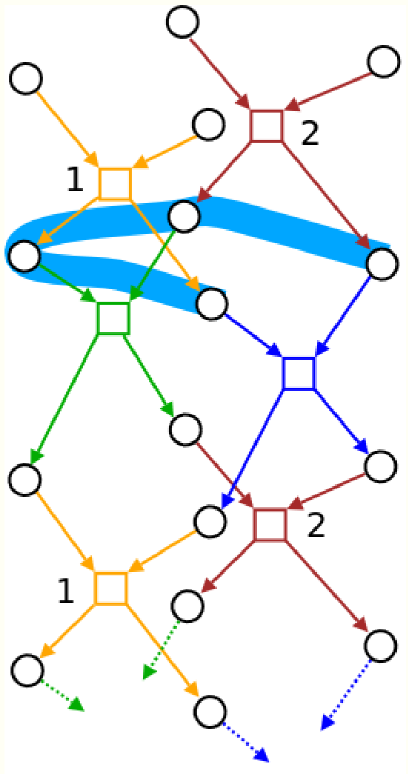




Occurrence Nets as processes of Net Systems

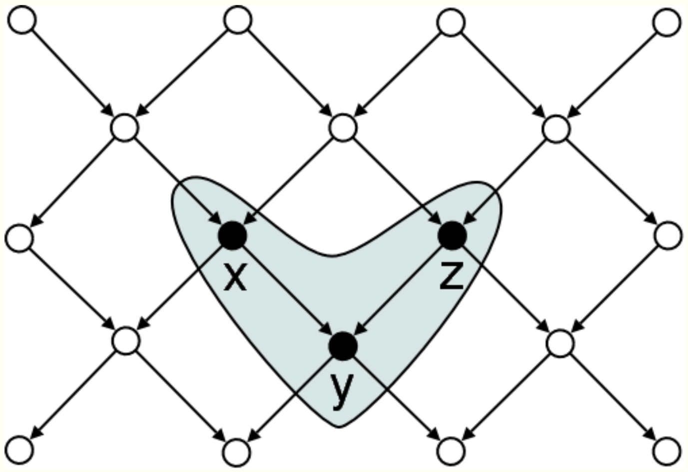
Case after 1 and 2





A closure operator based on li

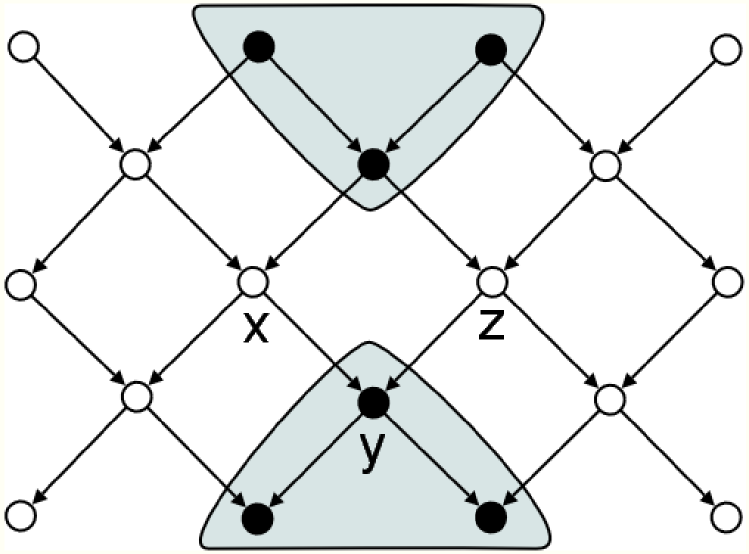
$(X, \leq), \qquad S \subseteq X$



A closure operator based on li

$(X, \leq), \qquad S \subseteq X$

$S^\star \;=\; \{x \in X \mid \forall y \in S : (x, y) \in li\}$



A closure operator based on li

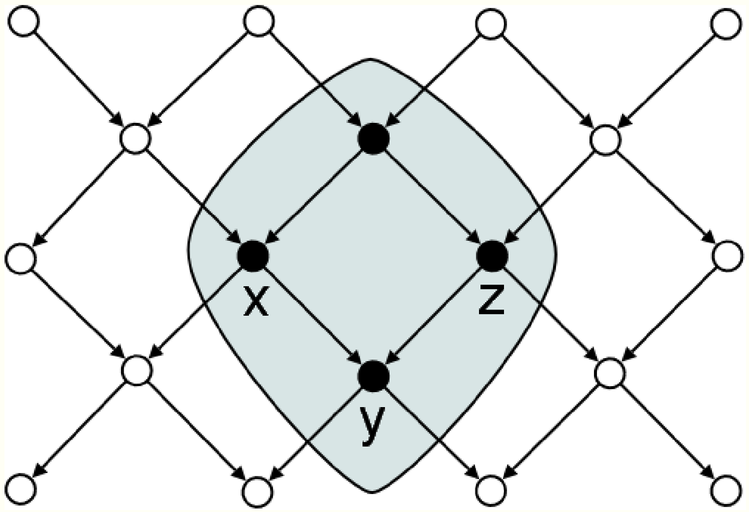
$(X, \leq), \qquad S \subseteq X$

$S^{\star} \; = \; \{x \in X \mid \forall y \in S : (x, y) \in li\}$

$$\boxed{(.)^{\star\star} : \mathbb{P}(X) \rightarrow \mathbb{P}(X)}$$

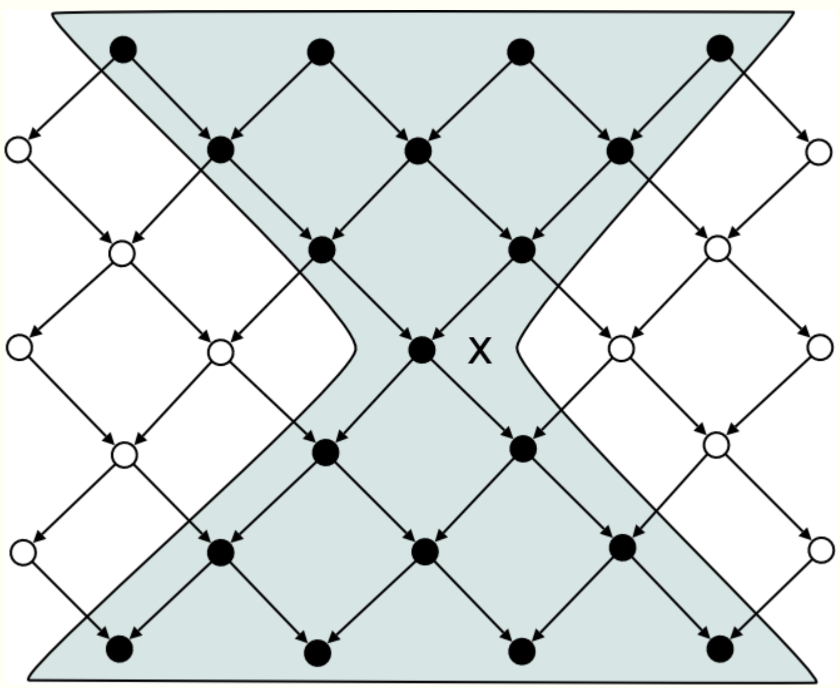
$S = S^{\star\star} \Leftrightarrow S$ is closed

$L(X)$ family of closed sets of X

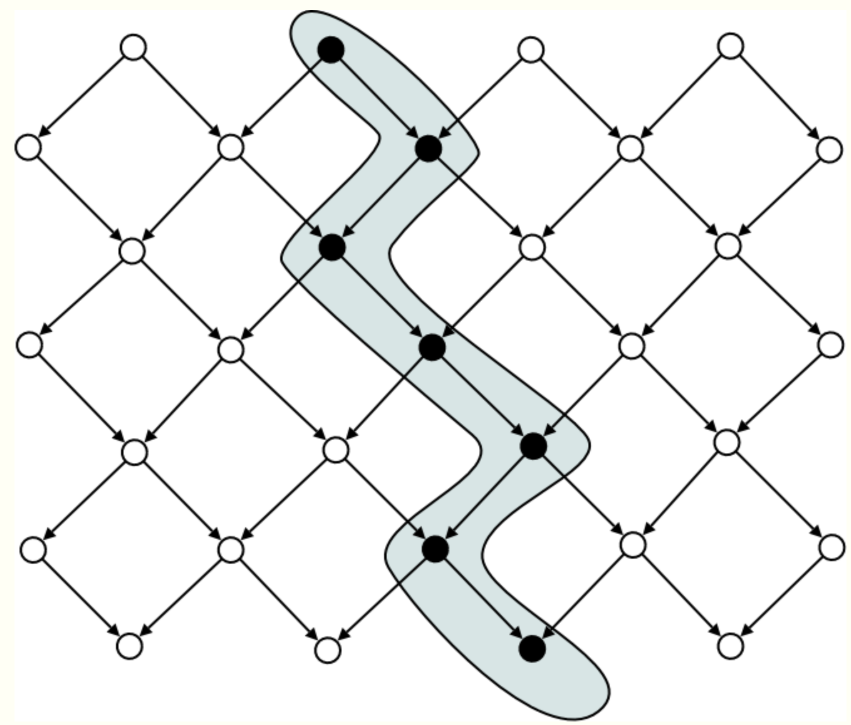


Examples of closed sets

cone ($cone(x) = \{y \in X \mid x \text{ li } y\}$)

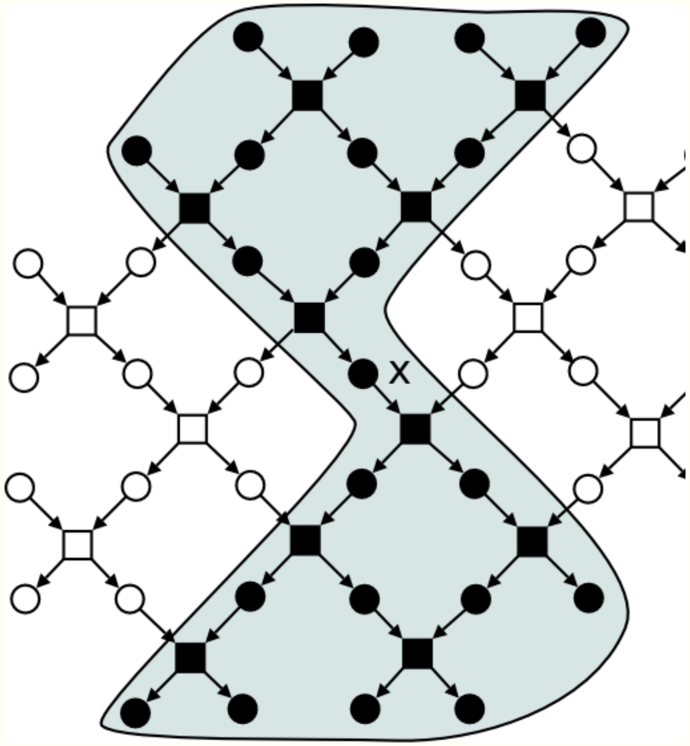


line (maximal clique of li)

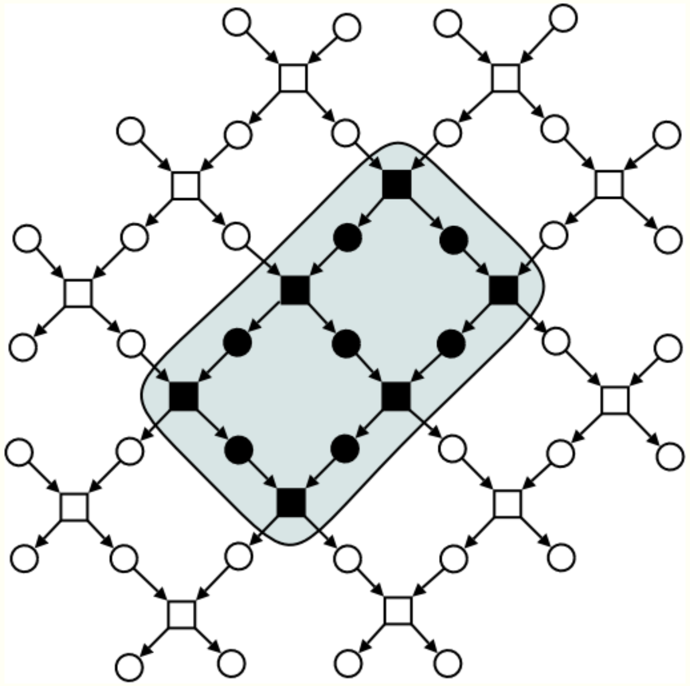


Examples of closed sets in Occurrence Nets

cone



diamond



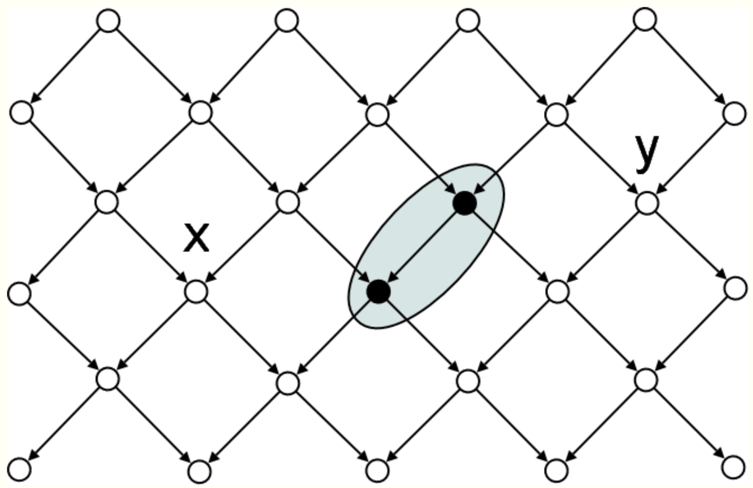
Intervals in space

x co y then $z \in [x, y]_{\mathrm{co}}$ if:

(i) $x \;\; \mathrm{co} \;\; z \;\; \mathrm{co} \;\; y$;

(ii) $\forall w : w < x, w < y \Rightarrow w < z$;

(iii) $\forall w : x < w, y < w \Rightarrow z < w$.



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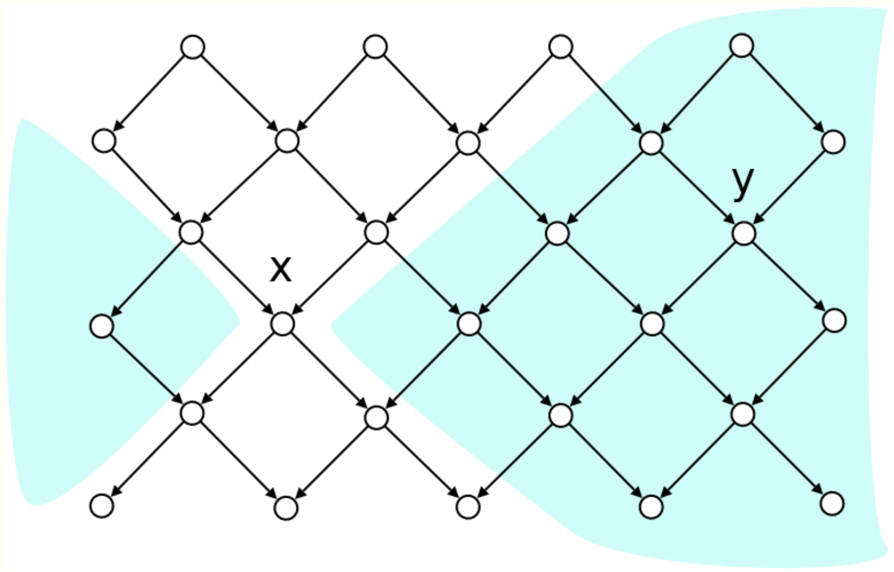

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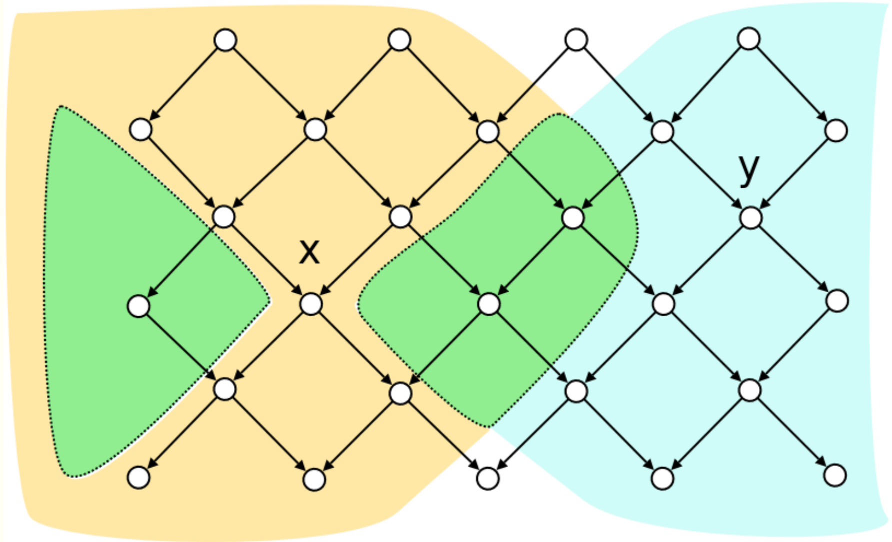

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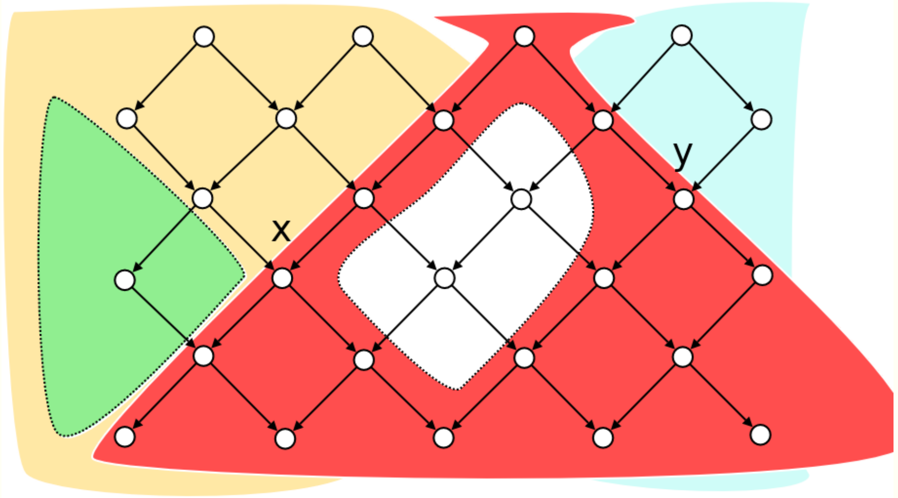

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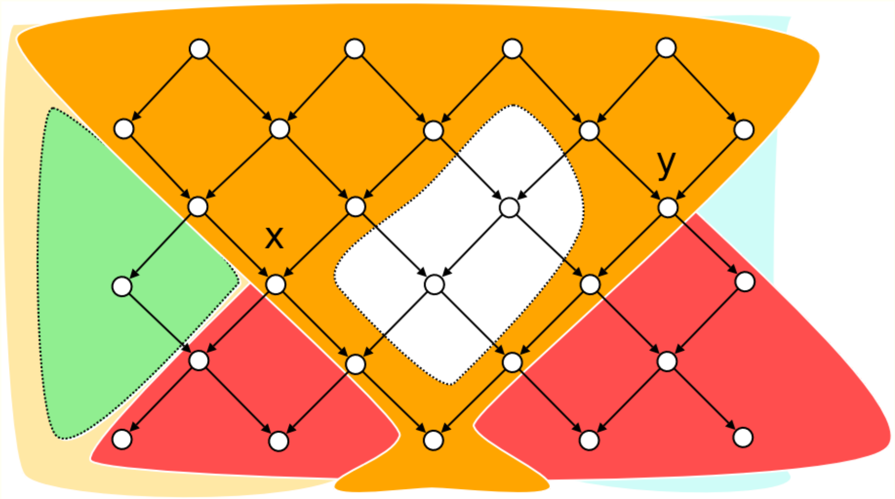


Closed sets are spatially convex

Globally

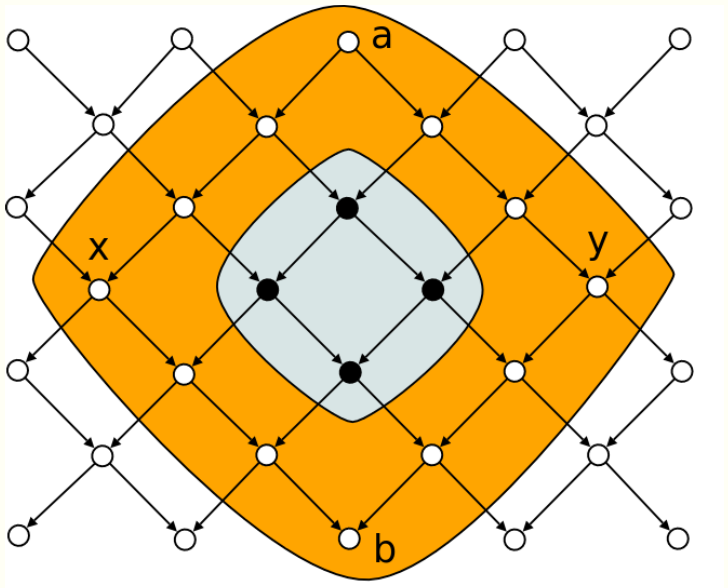
$A \in L(X), \quad x, y \in A, \quad x \text{ co } y;$

then $\quad [x, y]_{\text{co}} \subseteq A.$

Locally

$x, y, z \in X, \quad x \text{ co } z \text{ co } y, \quad z \in \{x, y\}^{\star\star};$

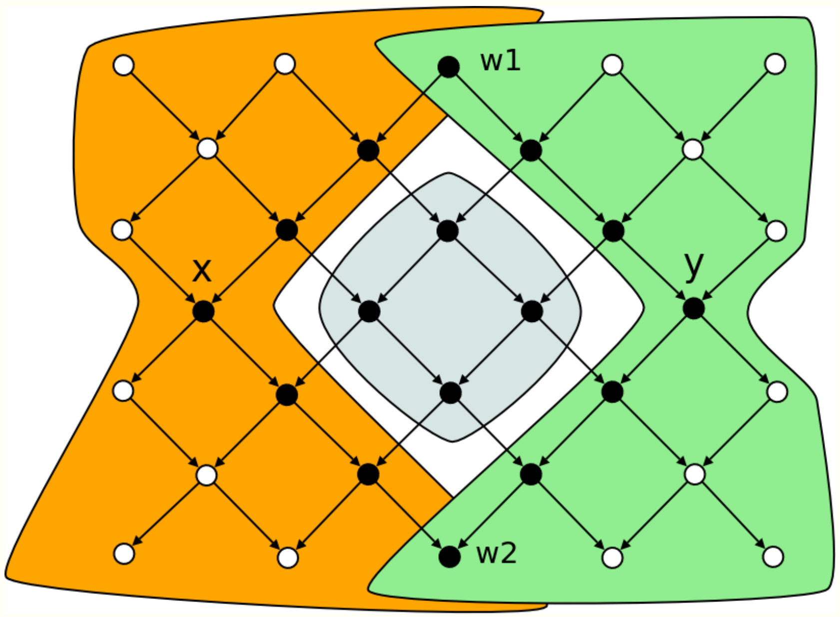
iff $\quad z \in [x, y]_{\text{co}}.$



Closed sets and spatially closed sets

$(X, \leq)$, $S \subseteq X$ is *spatially closed* iff $\forall x, y \in S \quad \forall z \in X$:

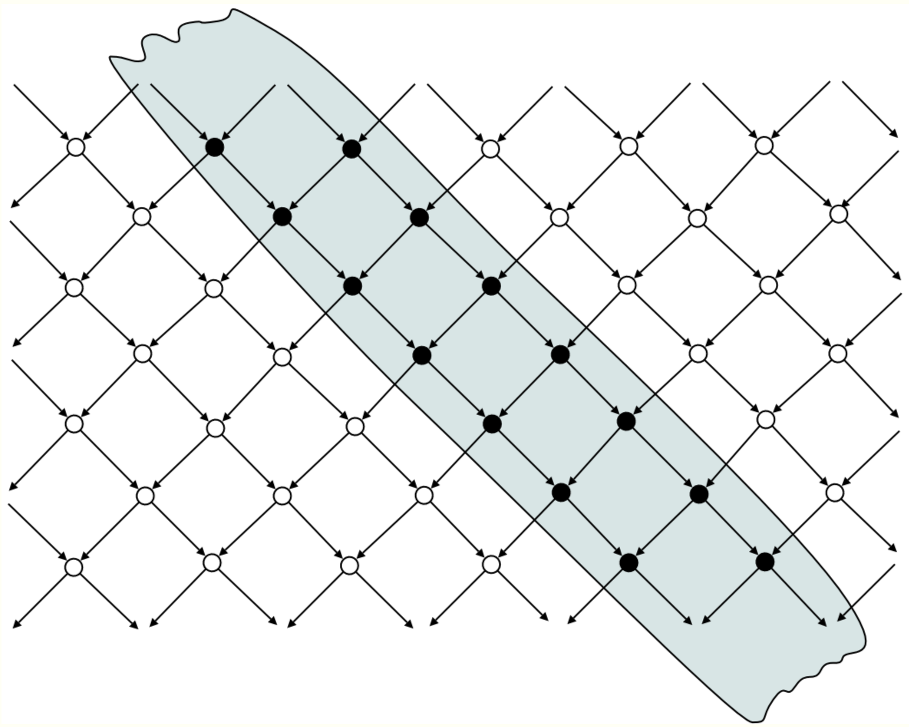
$$\forall w \in cone(x) \cap cone(y) \quad z \text{ li } w \Rightarrow z \in S$$



$S \in L(X) \Rightarrow S$ is spatially closed.

Closed sets and spatially closed sets

S is spatially closed $\not\Rightarrow$ $S \in L(X)$



A is spatially closed

$A^{\star} = \emptyset$

$A^{\star\star} = X$.

Summary of results on closure operators

$$\overset{\text{li}}{}$$

$$S^\star = \{x \in X \mid \forall y \in S : x \text{ li } y\}$$

$SC(X)$ family of spatially closed sets

$SC(X)$ are spatially convex

$SC(X)$ is a complete (algebraic) lattice

Conjecture: for K-dense posets $SC(X) = L_{\text{li}}(X)$.

Summary of results on closure operators

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co

$$S^\perp = \{\, x \mid \forall y \in S : x \, co \, y \,\}$$

$L_{\text{co}}(X) = (L_{\text{co}}(X), \subseteq, \emptyset, X, (.)^\perp)$ is a complete orthocomplemented lattice.

If X is N-dense, then $L_{\text{co}}(X)$ is orthomodular.

In occurrence nets: closed sets are *causally closed* subprocesses.

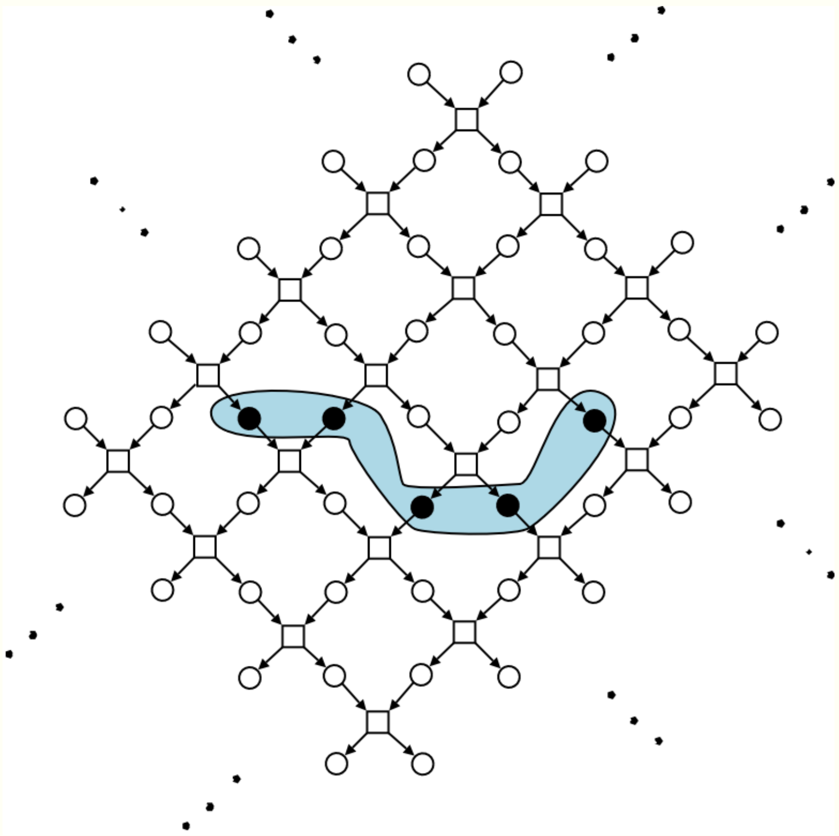
Causally closed subprocesses

S convex (w.r.t. li)

$e \in S \Rightarrow \bullet e \bullet \subseteq S$

$\bullet e \subseteq S \Rightarrow e \in S$

$e \bullet \subseteq S \Rightarrow e \in S$



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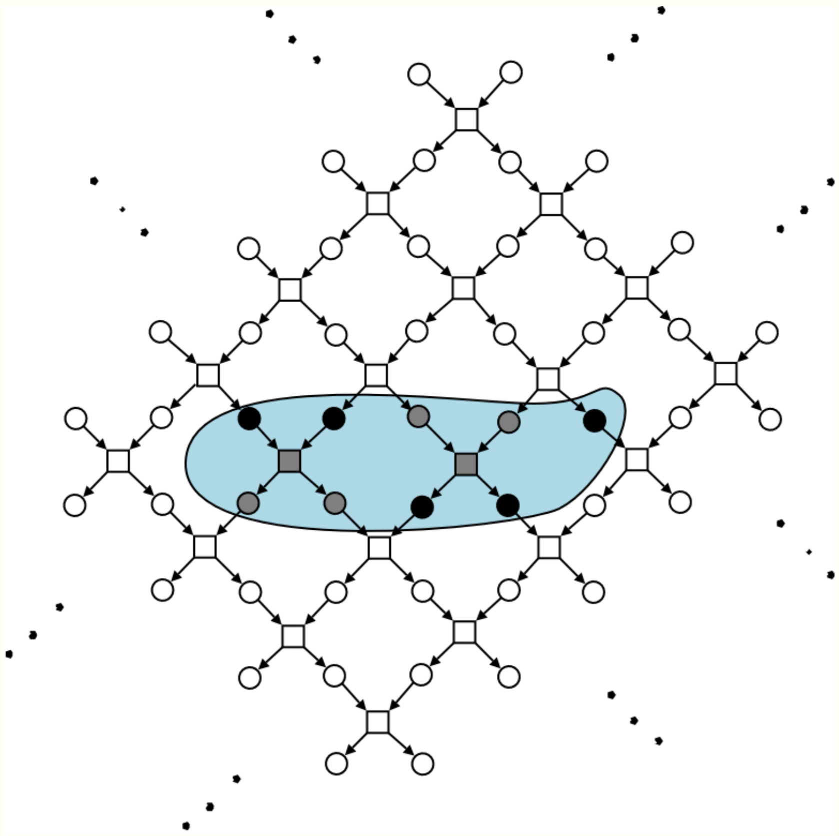

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