Automaticity of rational functions

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It is a well known fact that for a rational function $f(x) = \sum_{n=0}^{\infty} s_n x^n = \frac{p(x)}{q(x)}$ with coefficients s_n in a finite field the sequence $(s_n)_{n\in\mathbb{N}}$ is ultimately periodic. Therefore the sequence is k–automatic for all $k \geq 2$. (About k–automaticity see e.g. [1].)

In the following a rational function will be called k–automatic if and only if the sequence defined by its coefficients is k–automatic. Moreover a sequence (rsp. function), which is k-automatic for all integers $k \geq 2$, will simply be called automatic.

For two–dimensional sequences defined by rational functions

$$R(X,Y) = \sum_{n,m=0}^{\infty} s_{nm} X^n Y^m = \frac{P(X,Y)}{Q(X,Y)}$$

with coefficients s_{nm} in a finite field the situation is much more complicated. The description of automaticity we have here is not as simple as in the one–dimensional case.

From [4] we learn that a sequence $\mathbf{s} : \mathbb{N}^2 \to A$ is automatic if and only if each $M_a = \mathbf{s}^{-1}(a)$ is a rational subset of the monoid $\mathbb{N}^2$. These rational sets give a grid–like pattern on $\mathbb{N}^2$, which is similar to the notion of periodicity we have in one dimension.

The next theorem gives a characterization of those rational functions $R(X,Y) \in \mathbb{Z}_m[X,Y]$, where $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, which are automatic.

Theorem 1 *Let $m \geq 2$ be an integer. A rational function $R(X,Y) = \frac{P(X,Y)}{Q(X,Y)}$ with $P(X,Y)$, $Q(X,Y) \in \mathbb{Z}_m[X,Y]$ and $Q(0,0) = 1$ is automatic if and only if there exists a polynomial $S(X,Y) \in \mathbb{Z}_m[X,Y]$ such that $Q(X,Y)S(X,Y) = \prod(1 - \delta_i(X,Y))$, where the $\delta_i(X,Y)$ are finitely many monomials.*

This characterization together with the following two Lemmata provides a tool to generalize the results about the automaticity of the Pascal triangle, [2, 6], which is defined by the rational function $R(X,Y) = \frac{1}{1-Y(1+X)}$, and more general about the automaticity of orbits of linear cellular automata [3].

Let p be a prime number.

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Lemma 1 *Then* $R(X,Y) \bmod p^\alpha$ *is* p*–automatic for all* $\alpha \geq 1$.

Lemma 2 *If* $R(X,Y) \bmod p$ *is automatic, so is* $R(X,Y) \bmod p^\alpha$.

For the main theorem we need the following notation.
Notation Let $m \in \mathbb{N}$ and let p be a prime divisor of m, then we define

$$\begin{aligned} M_m(R(X,Y)) = \quad & \{p : \textit{there does not exist a polynomial } S(X,Y) \in \mathbb{Z}_p[X,Y] \\ & \quad \textit{with } S(X,Y)Q(X,Y) = \prod(1-\delta_i(X,Y)) \textit{ mod } p\}. \end{aligned}$$

Furthermore, denote by $\chi_m(R)$ the cardinality of $M_m(R(X,Y))$.

Theorem 2 *Let* $\chi_m(R)$ *and* $M_m(R(X,Y))$ *be defined as above, then for*

- $\chi_m(R) \geq 2$ *:* $R(X,Y) \bmod m$ *is not* k*–automatic for any* $k \geq 2$
- $\chi_m(R) = 1$ *:* $R(X,Y) \bmod m$ *is* p*–automatic, where* $p \in M_m(R(X,Y))$
- $\chi_m(R) = 0$ *:* $R(X,Y) \bmod m$ *is automatic.*

References

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