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Chaos in cellular automaton systems with majority rule *

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1. Introduction

The roughest, qualitative property which is expected to be elaborated when a complex dynamical system is taken into considerations, consists of predicting of a system long time behaviour. Systems which have the unique limit state and converge to it from any initial conditions may be said to forget everything, when time tends to infinity. In these cases the statistical analysis of time series, by dealing with time averages, provides an unique ergodic measure [1,2,3]. Dynamical systems usually exhibit some transient behaviour before they reach an asymptotic motion. In particular, if there exists more than one stationary state in the system, then the transient process can be extremely sensitive to the properties of an initial state and depending on properties of an initial state, is headed to a distinct stationary state. Cellular automata seems to be the simplest complex system for which this process can be observed

Generally, cellular automata can be seen as Markov chains— the systems of transition probabilities which converts any initial probability distribution defined in the space of lattice configurations into, so-called, the natural stationary measure [1,2,4]. The problem of construction of the probability distribution from a system of positive transition probabilities has been solved in *lattice* $\times$ *time* configuration space [3,5]— the stationary Markov chain is the limit Gibbs distribution with the Hamiltonian directly determined by transition probabilities. However, this description fails when a cellular automata local rule is of a deterministic type and also it does not give the measure on the lattice configuration space itself. It is known that the natural measure concentrates on "natural" configurations, corresponding to, so-called ground states [5], of the, in general, unknown Hamiltonian. Therefore, the deterministic cellular automata offer limit models for investigation the most fundamental problem of equilibrium statistical physics, which is to describe all Gibbs measures for a given Hamiltonian and observe relations between them and their coexistence.

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In particular, system of two-level spins occupying a square lattice, which locally interact with majority vote (*Centre-East-North* neighbours $\rightarrow$ *Centre* spin, called Toom model [6,7]) though its simplicity, allow to study the mentioned problem. This model is known [8] that any initial configuration is attracted to few, very regular configurations. Moreover, if an initial configuration is close to the maximal disordered configuration what means that each spin takes any of its states either *up* or *down* with probability $\frac{1}{2}$, then the critical behaviour is observed. It denotes that there is an uncertainty which of ground state configurations attracts a given initial configuration. Therefore, in the following we will concentrate on ground states attractor boundaries for the Toom model considered on a square lattice imposed with periodic boundary conditions. Using the modified box-counting method [4] we show the fractal character of the attractor boundary.

The probabilistic Toom model is equivalent to the equilibrium statistical mechanics of the Ising model on hcp [3] or fcc [9,10] three dimensional triangular lattice.

2. Fractal character of boundary

It appears that for each lattice size L there exists a p interval around $\frac{1}{2}$ where the considered model stabilises with any of following configurations:

— homogeneous with all spins up;

— homogeneous with all spins $down$;

— *mixed* configuration— the final pattern consists of two homogeneous regular clusters in stripe shape, going vertically, horizontally or along the West-South to North-East diagonal;

— *zigzag* configurations— there are two separated areas with different mixed configurations on a final pattern;

— there are three large areas on the lattice with different mixed states;

...

At $p = \frac{1}{2}$ probability that cellular automata is led to any of the first three types from the above list seems to be independent of lattice size and approximately stabilises at large L with $\frac{1}{3}$ [8]. There are no priori hints to which of these configurations the system is driven.

Let us say that our initial conditions are uncertain by the amount ε in the sense that when we say that the initial configuration is σ_0, what we really know is only that the initial configuration σ differs form σ_0 by one spin state. Thus the radius of initial uncertainty is

$$\varepsilon = |\sigma - \sigma_0| = (\frac{1}{L})^2 \tag{1}$$

We call an initial configuration σ_0 ε-certain if it yields all configurations distinct by ε to the same attractor [4]. An initial configuration σ_0 is ε-uncertain, if there exists a configuration which lies within a distance ε and which yields to different attracting point. In this case, one can say that the ε-uncertain configuration lies within a distance ε to the basin boundary.

For typical initial configurations the probability to obtain an ε-uncertain initial condition is the fraction of the volume of the configuration space which lies within ε distance to the boundary. Denote this fraction by $f(\varepsilon)$. It is known that [4] for a simple non fractal boundary $f(\varepsilon)$ scales linearly with ε, $f(\varepsilon) \sim \varepsilon$. However, when the boundary is fractal, then $f(\varepsilon)$ has the following scaling [4]:

$$f(\varepsilon) \sim \varepsilon^{\alpha} \tag{2}$$

where α, called the *uncertainty exponent* [4], is less than 1.

We estimate the uncertainty exponent α on the basis of the following experiment. For a given lattice size, L, we pick an initial configuration at random by tossing at each lattice site *up* spin state at the same probability p. Then we perturb the initial configuration by flipping one spin state and repeat it one-by-one along all sites of the lattice. So that, the uncertainty of initial configuration is (1) and we deal with all configurations within the initial uncertainty. We record which attractor (all spins *up*, all spins *down*, *mixed* or *zigzag* configuration) each of these initial configurations goes to. If all of them do not all go to the same attractor, then we count the original initial configuration as uncertain. We randomly choose 100–150 initial configurations for $p = 0.51 \pm 0.003$. Then we determine the fraction $\hat{f}(\varepsilon)$ of these that are uncertain. We repeat this experiment for different values of $\varepsilon : (\frac{1}{12})^2,\ (\frac{1}{24})^2,\ (\frac{1}{36})^2,\ (\frac{1}{48})^2,\ (\frac{1}{60})^2,\ (\frac{1}{72})^2$. According to [11, 12] we can expect that $f(\varepsilon)$ is approximately proportional to $\hat{f}(\varepsilon)$, so that α can be extracted from scaling $\hat{f}$ with ε. Fig.1 is the plot of the results on log-log axes. We have too little points to make the fit precise. However, we can learn about tendency whether or not α is lower than 1. The slope of the best line fit gives the power of ε as $\alpha \approx 0.014$ what denotes the " huge" fractality of the boundary.

There are some more regularities that have been observed in the performed experiments which allow to say more about relations between stationary configurations. They can be summarised as follows:

1. There was no single experiment where a configuration primary driving to *mixed* stationary configuration was ε-certain. There were always large areas in the initial configurations which were sensitive to the spin flip. There was only one experiment in which the configuration yielding all spins *down* occurred ε-certain. These observations were held independently of lattice size L. They suggest that initial configurations which provide all spins *downs* or *mixed* final configurations are ε-uncertain independently of lattice size. Initial configurations which led to all spins *up* stabilisation could appear either ε-certain or ε-uncertain. However, the probability that such a configuration is ε-certain depends on L, Fig.2.
2. We noticed a large discrepancy between numbers of unstable spins in ε-uncertain configurations, Fig.3. The unstable spins usually form kind of islands in the sea of stable spins. Anyway, the islands consisting of one isolated spin have been observed, too. Moreover, a single spin flip in

any initial configuration which unperturbed led the system to the *mixed* stationary configuration, provided the stabilisation at either any of homogeneous configuration or a mixed configuration, however *always* with the same geometric properties. For example, initial patterns with two vertical stripes were changed to all spins *up*, or all spins *down*, or two vertical stripes but with the different from original size. On the other hand, the single spin perturbation of a configuration which unperturbed led to any homogeneous configuration could effect all geometric types of *mixed* stationary configurations, but the opposite homogeneous state had not been observed.

3. The most uncertain initial configuration observed in our experiments was the configuration originally leading to *zigzag* stabilisation. However, we observed *zigzag* stationary configuration only twice— once at $L = 36$ and once at $L = 60$.

3. Closing remarks

In the following we found that typical random spin configurations initially distinct by a single spin state can provide different stationary configurations. Our rough estimation for the uncertainty parameter α of the boundary between basins of attractors provides for α the number much less than 1 what suggests that the boundary between basins of attractors is fractal. Therefore, the observed final state sensitivity which makes inefficient any predictions to cellular automata stabilisation, cannot be improved by increasing the lattice size. However, as these properties are ruled by local fluctuations, one can expect that the chaotic behaviour should be less sensitive to one spin flip with the growing lattice size.

Moreover, because of observed distinct qualitative properties of stabilisation just because of a single spin perturbation, we strongly support the opinion that *mixed* and any other higher complex configurations are configurations of higher energy than the two basic homogeneous ones. We believe that so huge presence of the stabilisation with these higher energy configurations can be connected with the geometric properties of boundaries in a space of periodic configurations on the infinite lattice. This extra non-local interaction which demands the complete adjustment of the final configurations to the boundaries, effects in lowering of energy contained in configurations with stripes in the proper shape. In consequence, the energy gap between the ground and some of the first level states becomes indistinguishable by experiment. We have some preliminary results that violating the square periodic boundaries, for example, be setting random size of each row, the final configurations space reduces drastically. In the mentioned example there were present only both homogeneous configurations and horizontal stripes configurations.

It has been proved rigorously [5,13] that even in the thermodynamic limit when the lattice period grows up to infinity the stationary state somehow remembers its periodic boundary conditions .

Since the famous two-dimensional Ising model, that is the simplest lattice model which deals with ferromagnetic interactions, which is governed by the Hamiltonian

$$\mathcal{H} = -\mathcal{J} \sum_{|i-j|=1} \sigma_i \sigma_j , \qquad \mathcal{J} > 0 \tag{3}$$

yields two homogeneous ground states, we are closing with the conclusion that the Hamiltonian (3) can be considered as the first approximation for the energy contained in the finite Toom cellular automata model.

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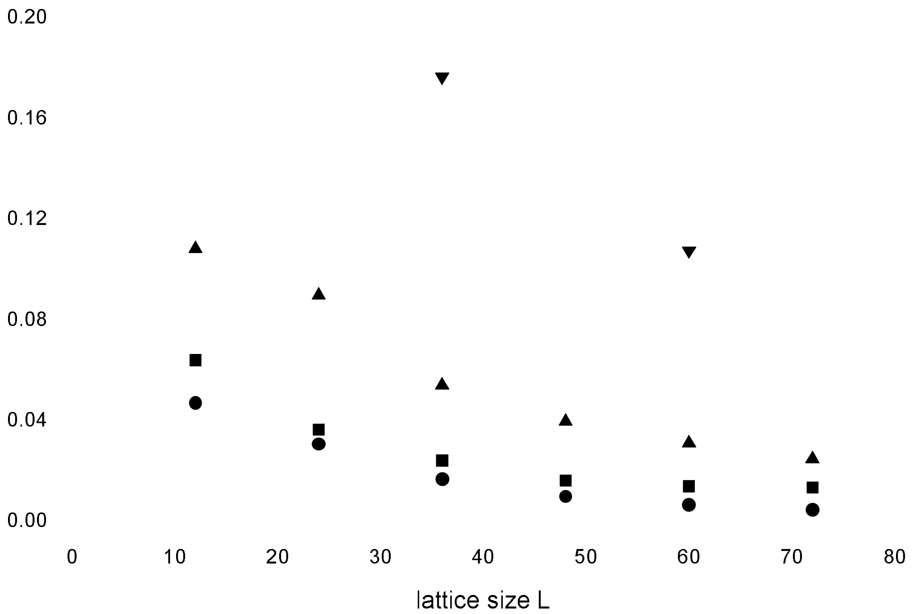

Fig.1: α estimation

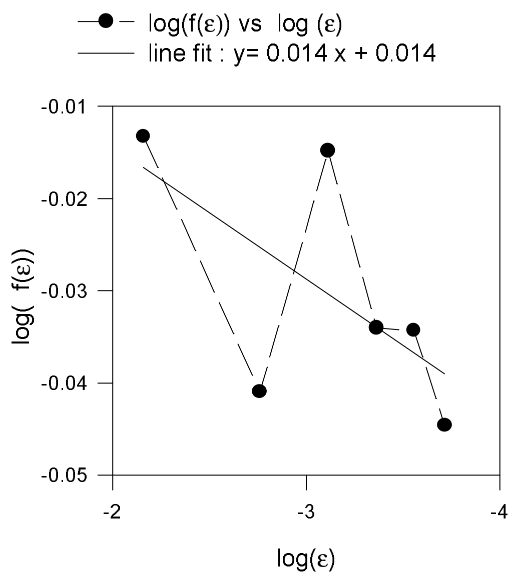

Fig.2: Fraction of ε–certain configuration among events with *up* stabilization

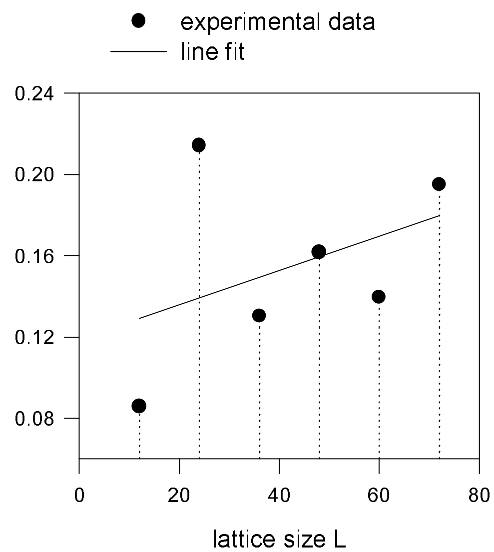

Fig.3: Fraction of unstable spins in ε–uncertain configuration (with errors)