Algebraical constructions of reversible generalized Pascal triangles and one-dimensional cellular automata

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Generalized Pascal triangles (GPT) are defined so that they immediately correspond to computations of one-sided one-dimensional cellular automata (1DCA) from finite initial configurations. (The mentioned automata have binary local transition functions and neighborhood types $(0,1)$ or $(-1,0)$.) Formally, GPT are associated to some finite algebras $\mathcal{A} = \langle\, \mathbf{A}; *, \mathrm{o}\rangle$ of signature $(2,0)$ and words $w \in \mathbf{A}^{+}$. $\mathrm{GPT}(\mathcal{A}; w)$ is constructed top-down from $\mathcal{A}$ and w similarly as the classical Pascal triangle is constructed from $\langle \mathbb{N}; +, 0\rangle$ and the constant 1. In the relation to one-sided 1DCA the binary operation of the underlying algebra corresponds to the local transition rule and its constant to the quiescent state. The relationship to other one-dimensional CA also can be established, although it is not so immediate.

A GPT will be called reversible if it also can be locally constructed bottom-up, using a binary operation. An algebra will be called reversible if all its GPT are reversible. Several kinds of reversibility of algebras will be introduced; all of them lead to reversible 1DCA. (However, general reversibility of 1DCA is not completely covered by them because it allows reverse local transition rules with more than two arguments.)

Some constructions of reversible algebras from groupoids satisfying cancellation laws (both or one of them) are presented. Elements of the constructed algebra belong to the cartesian square of the groupoid, and the constructed operation is determined by a formula $(x_1, x_0) * (y_1, y_0) = (T_1, T_0)$, where T_1, T_0 are suitable terms composed from x_i, y_i, and the groupoid operation symbol. Further, two constructions of reversible algebras from another reversible algebras will be given; we call them permutation of values and permutation of arguments. These constructions preserve base sets, and usually also the constants o of the algebras. (Hence only permutations of $\mathbf{A} \setminus \{\mathrm{o}\}$ are usually considered, and nontrivial results can be obtained only for $\mathrm{card}(\mathbf{A}) > 2$.) The constructions are very simple from the algebraic point of view but nevertheless they can substantially change the types of the corresponding GPT (and also corresponding 1DCA). For example, we can obtain an algebra with GPT containing all words from $\mathbf{A}^{+}$ in its rows from a semilinear algebra $\mathcal{A} = \langle\, \mathbf{A}; *, \mathrm{o}\rangle$ of degree 1. (We say that $\mathcal{A} = \langle\, \mathbf{A}; *, \mathrm{o}\rangle$ is a semilinear algebra of degree 1 if the set of rows of every of its GPT is a regular language.) The idea can be immediately applied also to the classical 1DCA (i.e., those with the neighborhood type $(-1,0,1)$), and also to higher-dimensional cellular automata.