Sand Dripping in Linear Space

Jérôme Olivier Durand-Lose[*][†]

LaBRI, URA CNRS 1 304,
Université Bordeaux I,
351, cours de la Libération,
33 405 Talence Cedex,
France.

e-mail: `jdurand@labri.u-bordeaux.fr`

Abstract

We study the evolution of an empty linear sand pile system which receives a sand grain in its first pile each time. We encode it with another cellular automata, the Chip Firing Game. Geometrical observations lead to a signal encoding of configurations. The interactions of signals are studied. Configurations are divided in two parts of slopes 1 and 2. We make asymptotic approximations of various parameters defining the configurations. Especially, we find that the configurations are expanding regularly in terms of the square root of the number of fallen grains and the ratio of the width of the two parts of the configurations is $\sqrt{2}$.

Sand Piles Model (SPM) and Chip Firing Games (CFG) are very simple systems based on local interactions which conserve the total number of grains. In SPM, a grain moves to a neighbor site if the number of grains in the two piles differs by more than a given threshold, whereas in CFG a site gives a chip to each of its neighbors if its number of chips is above the number of neighbors. In the linear case, both games are 1-D cellular automata and are equivalent via a simple encoding [7].

These models were first introduced in physics to study the so called '$1/f$ phenomenon' and 'Self-Organized Criticality' in avalanches, earthquakes, plasmas, … [12].

SPM and CFG, like Petri Nets [10], are used to model parallel computing, in particular the flows of informations in a system. Grains model data or tasks, and piles, a processor network [11]. The goal is to find a simple, fast and costless local reorganization which ensures that all processors will have almost the same amount of work.

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[†]This research was done while the author was in the Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Santiago, Chile.

We limit the problem to linear space. Goles and Kiwi have widely studied the case of linear SPM and the related CFG [6, 7]. They have detailed the dynamic as well as demonstrated the unicity of the final configuration whatever the order of the iterations is for various sequential cases.

Anderson *et al.* studied a similar problem in [1]. They also found signals in CFG, but their model allows that more than one grain travel between two piles at the same time.

We study the parallel updating of a system which is empty at start and that, while relaxing, receives grains in its first site. It can be understood as sand dripping in a thin but long hourglass.

Triangle straps appear on 3-D representations of the evolution of the system. Using a height difference notation, we switch to the CFG model. All configuration are then of the form $2^* \, (13)^* \, (02)^* \, 1^*$. They are made of two parts of distinct slopes (2 for $2^* \, (13)^*$ then 1 for $(02)^* \, 1^*$). Each one is itself composed of two sections of different motifs.

The configurations are parted in four sections amounting to the four motifs, delimited by signal-like frontiers. The first signals, between 2^* and $(13)^*$, and last one, between $(02)^*$ and 1^*, are always moving between the borders of the configurations and the second one, between $(13)^*$ and $(02)^*$. The second signal is fixed, it is only moved by the two other ones. It somehow counts the comming and going of the two others.

We give asymptotic approximations of the different parameters and geometrical shape. As expected, since the grains must fill a surface, we found that the height and width of the system are growing in $o(\sqrt{n})$, where n is the number of fallen grains. Moreover the relative length of the two parts converges towards $\sqrt{2}$.

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