# Cellular automata, finite automata and the Schur property for polynomials

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Although they have similar names, finite automata and cellular automata are quite different. It is nevertheless interesting to ask whether patterns generated by the time evolution of a one-dimensional cellular automaton can be also generated by two-dimensional finite automata. This question has been addressed in the case of (one-dimensional) linear cellular automata by Korec [4] and by F. von Haeseler, H.-O. Peitgen, A. Petersen and the authors [1, 2]. One useful tool is the "$m$-Fermat property": a polynomial on a ring $R$ is called $m$-Fermat if

$$P^m(X) = (P(X))^m.$$

In particular if $R = \mathbb{Z}/p\mathbb{Z}$ with $p$ prime, every polynomial is $p$-Fermat and, as a consequence,

$$\forall t \geq 0,\ \forall r \in [0, p-1],\ P^{pt+r}(X) = P^t(X^p)P^r(X).$$

The generalization we study here consists in replacing the sequence $(P^t(X))_{t\geq 0}$ which represents the time evolution of a one-dimensional linear cellular automaton by a sequence of polynomials $(Q_t(X))_{t\geq 0}$ verifying what we call "the $p$-Schur property":

$$\forall t \geq 0,\ \forall r \in [0, p-1],\ Q_{pt+r} = Q_t(X^p)Q_r(X).$$

If $Q_t(X) = \sum a_{t,n}X^n$ and $\deg Q_t \leq t$, this is equivalent to saying that:

$$(*)\ \ \forall t, n \geq 0,\ \forall i, j \in [0, p-1],\ a_{pt+i,pn+j} = a_{t,n}a_{i,j}.$$

Of course the sequence of polynomials defined by $Q_t(X) = (1 + X)^t$ satisfies the $p$-Schur property on the ring $\mathbb{Z}/p\mathbb{Z}$, the corresponding sequence $(a(t, n))_{t,n}$ is nothing but the sequence $(\binom{t}{n})_{t,n}$ and the property $(*)$ is the Lucas theorem. Analogously the sequence of polynomials $Q_t(X) = B^t(X)$

where $B$ is a fixed polynomial has the $p$-Schur property on $\mathbb{Z}/p\mathbb{Z}$. We can hence generalize (one-dimensional) linear cellular automata by taking a sequence of polynomials having the $p$-Schur propery instead of the sequence $B^t(X)$. Another (less evident) example of a $p$-Schur sequence of polynomials is given by the Legendre polynomials: *the Legendre polynomials have the $p$-Schur property for any odd prime $p$*, this has been proved by Schur (according to [3, 6]). We also note that sequences $(a(t,n))_{t,n}$ having the property $(*)$ have been studied by McIntosh [5].

We have obtained results on the Legendre, Hermite and Tshebishev polynomials using *inter alia* results of Carlitz. For example, for the Legendre polynomials, using the $p$-Schur property and the relation (see [7] for example)

$$\sum_{t\geq 0} P_t(X)Y^t = (1-2XY+Y^2)^{-\frac{1}{2}}$$

we can prove the following. *Define the sequence $(a(t,n))_{t,n}$ by $P_t(X) = \sum a_{t,n}X^n$ where $P_t$ is the $t$-th Legendre polynomial. Then the sequence $(a(t,n))_{t,n} \bmod \mathrm{d}$ is $p$-automatic for $d = p^s$ where $p$ is an odd prime and not $k$-automatic if $d$ has two different odd prime divisors.*

## References


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