



# Cellular Automata with Memory

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REFERENCES

<u>Conventional</u> CA are Markovian (ahistoric, memoryless<sup>1</sup>): The next state of a cell depends solely on its current neighborhood ( $\mathcal{N}$ ) configuration.

$$\sigma_i^{(T+1)} = \phi(\{\sigma_{j\in\mathcal{N}_i}^{(T)}\}) \quad \forall i$$

CA with embedded memory

$$\begin{split} \mathbf{s}_{i}^{(T)} &= \mathbf{s}\left(\sigma_{i}^{(1)}, \dots, \sigma_{i}^{(T-1)}, \sigma_{i}^{(T)}\right) \quad \rightarrow \quad \sigma_{i}^{(T+1)} = \phi\left(\left\{\mathbf{s}_{j\in\mathcal{N}_{i}}^{(T)}\right\}\right) \quad \forall i \\ f_{i}^{(T)} &= \phi\left(\left\{\sigma_{j\in\mathcal{N}_{i}}^{(T)}\right\}\right) \quad \rightarrow \quad \sigma_{i}^{(T+1)} = \mathbf{s}\left(f_{i}^{(1)}, \dots, f_{i}^{(T-1)}, f_{i}^{(T)}\right) \quad \forall i \\ \mathbf{CA with \ delay \ memory} \end{split}$$



Extension to the standard framework where:

The mapping  $\phi$  remains unaltered, every cell retains historic memory of its past states by means of the trait state s. So to say, cells canalize memory to the map.

<sup>1</sup> 
$$\sigma_i^{(T)} = \phi(\{\sigma_{j\in\mathcal{N}_i}^{(T-1)}\})$$
 Chain like (indirect) effect of the past  $\equiv$  memory-one

Example

 $\phi$ : cell alive if any cell in its neighborhood is alive (speed of light).



s: Majority (most frequent, mode) memory. Last state in case of a tie. Embedded:  $s_i^{(T)} = mode(\sigma_i^{(1)}, \dots, \sigma_i^{(T)}) \rightarrow \sigma_i^{(T+1)} = \phi(\{s_{i \in \mathcal{N}_i}^{(T)}\})$  $\forall i$  $\sigma$ ssssss $\sigma$  $\phi$ Ф **Delay:**  $f_i^{(T)} = \phi(\{\sigma_{i \in \mathcal{N}_i}^{(T)}\}) \rightarrow \sigma_i^{(T+1)} = mode(f_i^{(1)} \dots, f_i^{(T)})$  $\forall i$ 

**MAJORITY** memory  $\Rightarrow$  **INERTIAL** effect

#### <u>LIMITED TRAILILING</u> memory of the last $\underline{\tau}$ states:

Example:  $\tau = 3$ -Majority memory:  $\mathbf{s}_{i}^{(T)} = mode \left(\sigma_{i}^{(T-2)}, \sigma_{i}^{(T-1)}, \sigma_{i}^{(T)}\right) \qquad \sigma_{i}^{(T+1)} = mode \left(f_{i}^{(T-2)}, f_{i}^{(T-1)}, f_{i}^{(T)}\right) \qquad T > 2$ Speed of light with  $\tau = 3$ -Majority Embedded memory  $\boldsymbol{\sigma}$ 

1D Parity rule  $\phi$ : cell alive iff odd number of alive cells in its neighborhood



Elementary Legal Rules with Majority Memory



#### Weighted memory (unlimited trailing embedded memory)

$$m_i^{(T)}(\sigma_i^{(1)}, \dots, \sigma_i^{(T)}) = \frac{\sigma_i^{(T)} + \sum_{t=1}^{T-1} \alpha^{T-t} \sigma_i^{(t)}}{\prod_{t=1}^{T-1} \alpha^{T-t}} \equiv \frac{\omega_i^{(T)}}{\Omega(T)} = \frac{\sigma_i^{(T)} + \alpha \omega_i^{(T-1)}}{1 + \alpha \Omega(T-1)} *$$

$$\begin{bmatrix}
 \sigma_i^{(T-2)} \\
 \sigma_i^{(T-1)} \\
 \sigma_i^{(T)} \\
 \times \alpha
 \end{bmatrix}
 \times \alpha$$

 $\nabla \alpha T - 1$ 

(1)

The choice of the **memory factor**  $0 \le \alpha \le 1$  fits the memory effect: the limit case  $\alpha = 1$  is equivalent to unlimited trailing *majority* memory, whereas  $\alpha << 1$  intensifies the contribution of the most recent states (short-range memory). The choice  $\alpha = 0$  leads to the ahistoric model.

If  $\sigma \in \{0, 1\}$ , trait state s by rounding the weighted mean m:

$$s_i^{(T)} = \begin{cases} 1 & if \quad m_i^{(T)} > 0.5 \\ \sigma_i^{(T)} & if \quad m_i^{(T)} = 0.5 \\ 0 & if \quad m_i^{(T)} < 0.5 \end{cases} \qquad s_i^{(1)} = \sigma_i^{(1)} \ , \ s_i^{(2)} = \sigma_i^{(2)}$$

Implementation

k = 2:  $\alpha$ -MEMORY EFFECTIVE if  $\alpha > 0.5$ \*{ $\sigma_i^{(t)}, t = 1, 2, ..., T$ } NO NEEDED

**Drawback:**  $\alpha \rightarrow$  real numbers

#### Weighted memory (unlimited trailing delay memory)

$$m_i^{(T)}(f_i^{(1)}, \dots, f_i^{(T)}) = \frac{f_i^{(T)} + \sum_{t=1}^{T-1} \alpha^{T-t} f_i^{(t)}}{1 + \sum_{t=1}^{T-1} \alpha^{T-t}} \equiv \frac{\omega_i^{(T)}}{\Omega(T)} = \frac{f_i^{(T)} + \alpha \omega_i^{(T-1)}}{1 + \alpha \Omega(T-1)} *$$

$$\begin{array}{c} f_i^{(1)} & \times \alpha^{T-1} \\ & \ddots & \ddots \\ \hline f_i^{(T-2)} & \times \alpha^2 \\ \hline f_i^{(T-1)} & \times \alpha \\ \hline f_i^{(T)} & \times 1 \end{array}$$

The choice of the **memory factor**  $0 \le \alpha \le 1$  fits the memory effect: the limit case  $\alpha = 1$  is equivalent to unlimited trailing *majority* memory, whereas  $\alpha << 1$  intensifies the contribution of the most recent states (short-range memory). The choice  $\alpha = 0$  leads to the ahistoric model.

If  $\sigma \in \{0, 1\}$ , trait state s by rounding the weighted mean m:

$$s_i^{(T)} = \begin{cases} 1 & if \quad m_i^{(T)} > 0.5 \quad \equiv 2\omega_i^{(T)} > \Omega(T) \\ f_i^{(T)} & if \quad m_i^{(T)} = 0.5 \quad \equiv 2\omega_i^{(T)} = \Omega(T) \\ 0 & if \quad m_i^{(T)} < 0.5 \quad \equiv 2\omega_i^{(T)} < \Omega(T) \end{cases}$$

Implementation

k = 2:  $\alpha$ -MEMORY EFFECTIVE if  $\alpha > 0.5$ \*{ $f_i^{(t)}, t = 1, 2, ..., T$ } NO NEEDED

**Drawback:**  $\alpha \rightarrow$  real numbers

function cam T=8;SR=254;alpha=1.0;N=2\*T+1; [srb]=binarynumber(SR);left=[N 1:N-1];right=[2:N 1]; for memo=1:2 [SIGMA,OMEGA,omega]=init(T,N,alpha); switch memo case 1 % Embedded for t=1:T SIGMAH(t,:)=SIGMA;S=SIGMA; omega=(alpha\*omega)+SIGMA;OMEGAX=OMEGA(t); for i=1:N if(2\*omega(i)>OMEGAX)S(i)=1;end; % memory if(2\*omega(i)<OMEGAX)S(i)=0;end</pre> end [SIGMA]=RULE(S,N,srb,left,right); % rule HS(t,:)=S;end case 2 % Delay [SIGMA,OMEGA,omega]=init(T,N,alpha);S=SIGMA; for t=1:T SIGMAH(t,:)=SIGMA; [S] =RULE(SIGMA,N,srb,left,right); % rule SIGMA=S;HS(t,:)=S; omega=(alpha\*omega)+SIGMA;OMEGAX=OMEGA(t); for i=1:N if(2\*omega(i)>OMEGAX)SIGMA(i)=1;end % memory if(2\*omega(i)<OMEGAX)SIGMA(i)=0;end endend end subplot(3,2,2\*(memo-1)+1);image(33\*SIGMAH]);axis image;axis('off'): if(memo==1)title('sigma embedded');else;title('sigma delay');end subplot(3,2,2\*(memo-1)+2);imagesc(33\*HS,[0,44]);axis image;axis('off'); if(memo==1)title('s');else;title('f');end end print camembedelay.eps -depsc function [SIGMA,OMEGA,omega]=init(T,N,alpha); SIGMA(1:N)=0; SIGMA((N+1)/2:(N+1)/2)=1; OMEGA(1)=1.0;omega(1:N)=0; for t=2:T;OMEGA(t)=1+alpha\*OMEGA(t-1);end function [SIGMA]=RULE(S,N,srb,left,right); for i=1:N SIGMA(i)=srb(8-(4\*S(left(i))+2\*S(i)+S(right(i)))); end function [BN] =binarynumber(rule); BN(1:8)=0; irtx=rule; for ix=1:8 rest=mod(irtx,2);ratio=(irtx-rest)/2;BN(8-ix+1)=rest;irtx=ratio;

end





sigma delay



Elementary Legal Rules with  $\alpha$ -Memory

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Ahistoric	0.6	0.7	0.8	0.9	1.0	Ahistoric 0.6 0.7 0.8 0.9
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Rule 54	A	Å	Å	\$\$\$	-	Rule 54
Rule 94	᠕	᠕	᠕	⋒	A	
Rule 120		A		â		
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Rules 222,254						

#### Elementary legal rules with <u>low</u> $\alpha$ -delay memory



## Elementary, Legal Rules with $\alpha$ -delay Memory



## black = Damage spreading

Alonso-Sanz, R. (2013). Elementary cellular automata with memory of delay type. LNCS, 8155, 67-83.

# ECA with $\alpha$ -delay Memory



#### black = Damage spreading

<u>1D r=2</u> CA : neighborhood nearest and next-nearest neighbors

$$\sigma_i^{(T+1)} = \phi \big( \sigma_{i-2}^{(T)}, \sigma_{i-1}^{(T)}, \sigma_i^{(T)}, \sigma_{i+1}^{(T)}, \sigma_{i+2}^{(T)} \big)$$

In <u>totalistic</u> r = 2 rules:

$$\sigma_i^{(T+1)} = \phi \left( \sigma_{i-2}^{(T)} + \sigma_{i-1}^{(T)} + \sigma_i^{(T)} + \sigma_{i+1}^{(T)} + \sigma_{i+2}^{(T)} \right)$$

With memory:

$$\begin{split} \sigma_i^{(T+1)} &= \phi \big( s_{i-2}^{(T)} + s_{i-1}^{(T)} + s_i^{(T)} + s_{i+1}^{(T)} + s_{i+2}^{(T)} \big) \\ f_i^{(T)} &= \phi \big( \sigma_{i-2}^{(T)} + \sigma_{i-1}^{(T)} + \sigma_i^{(T)} + \sigma_{i+1}^{(T)} + \sigma_{i+2}^{(T)} \big) \end{split}$$

Totalistic k=r=2 rules are characterized by a sequence of binary values  $(\beta_s)$  associated with each of the six possible values of the sum (s) of the neighbors:

$$(\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)_{binary} \equiv \sum_{s=0}^{5} \beta_s 2^s = R \in [0, 63]$$
  
# totalistic r=2 rules =  $VR_2^6 = 2^6 = 64$ 

1D r=2 CA with  $\alpha$ -Embedded Memory



1D r=2 CA with Majority Embedded Memory



#### 1D r=2 CA with Majority Delay Memory



#### The 2D PARITY rule with Memory. Moore N.

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Alonso-Sanz, R., Martin, M. (2002). Cellular Automata with Memory: patterns starting with a single site seed. IJMPC, 13, 1.

# The 2D PARITY rule with low memory: $\alpha = 0.501$

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## $Memory: Chaos (III) \rightarrow Complex (IV)$

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- Martinez,G.J.,Adamatzky,A.,Alonso-Sanz,R.(2012). Complex dynamics of elementary cellular automata emerging from chaotic rules. *IJBC*,22,2.
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- Alonso-Sanz, R., Martin, M. (2005). One-dimensional cellular automata with memory in cells of the most recent value. CS, 15, 3.

Elementary Rules ( $\psi$ ) as Memory ( $\tau = 3$ ):

Embedded memory  

$$\begin{split} \mathbf{s}_{i}^{(T)} &= \psi\left(\sigma_{i}^{(T-2)}, \sigma_{i}^{(T)}, \sigma_{i}^{(T-1)}\right) \rightarrow \sigma_{i}^{(T+1)} = \phi\left(\left\{\mathbf{s}_{j\in\mathcal{N}_{i}}^{(T)}\right\}\right) \quad \forall i \\ f_{i}^{(T)} &= \phi\left(\left\{\sigma_{j\in\mathcal{N}_{i}}^{(T)}\right\}\right) \rightarrow \sigma_{i}^{(T+1)} = \psi\left(f_{i}^{(T-2)}, f_{i}^{(T)}, f_{i}^{(T-1)}\right) \quad \forall i \\ \mathbf{Delay memory} \end{split}$$

Example: 
$$\psi$$
= Majority  $\equiv$  ECA232  
 $s_i^{(T)} = mode\left(\sigma_i^{(T-2)}, \sigma_i^{(T)}, \sigma_i^{(T-1)}\right) \qquad \sigma_i^{(T+1)} = mode\left(f_i^{(T-2)}, f_i^{(T)}, f_i^{(T-1)}\right)$ 

Alonso-Sanz, R., Bull, L. (2009). Elementary cellular automata with elementary memory rules in cells: The case of linear rules. JCA, 1, 1. Alonso-Sanz, R. (2013). Elementary cellular automata with memory of delay type. JCA (in press).

The rule  $\phi = 150$  with delay ECA memories

+ 0	+ 1	+ 2	+ 3	+ 4	+ 5	+ 6	+ 7	+128	+129	+130	+131	+132	+133	+134	+135
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+ 8	+ 9	+10	+11	+12	+13	+14	+15	+136	+137	+138	+139	+140	+141	+142	+143
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+16	+17	+18	+19	+20	+21	+22	+23	+144	+145	+146	+147	+148	+149	+150	+151
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+32	+33	+34	+35	+36	+37	+38	+39	+160	+161	+162	+163	+164	+165	+166	+167
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+48	+49	+50	+51	+52	+53	+54	+55	+176	+177	+178	+179	+180	+181	+182	+183
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+56	+57	+58	+59	+60	+61	+62	+63	+184	+185	+186	+187	+188	+189	+190	+191
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+64	+65	+66	+67	+68	+69	+70	+71	+192	+193	+194	+195	+196	+197	+198	+199
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The rule 90 with embedded ECA memories



#### The rule 90 with embedded ECA memories



#### The rule 90 with embedded ECA memories





**<u>REVERSIBLE</u>** CA (Fredkin):  $\sigma_i^{(T+1)} = \phi(\{\sigma_{i \in \mathcal{N}_i}^{(T)}\}) \ominus \sigma_i^{(T-1)}$ **EMBEDDED MEMORY:**  $\sigma_i^{(T+1)} = \phi(\{s_{i \in \mathcal{N}_i}^{(T)}\}) \ominus \sigma_i^{(T-1)}$ **DELAY MEMORY:**  $\sigma_i^{(T+1)} = s(f_i^{(1)}, \dots, f_i^{(T)}) \ominus \sigma_i^{(T-1)}$  $\sigma_i^{(T-1)} = \phi(\{\sigma_{i \in \mathcal{N}_i}^{(T)}\}) \ominus \sigma_i^{(T+1)}$ **Reversion:**  $\sigma_i^{(T-1)} = \phi\left(\left\{\boldsymbol{s}_{i\in\mathcal{N}_i}^{(T)}\right\}\right) \ominus \sigma_i^{(T+1)} \qquad \sigma_i^{(T-1)} = \boldsymbol{s}\left(f_i^{(1)}, \dots, f_i^{(T)}\right) \ominus \sigma_i^{(T+1)}$  $\omega_i^{(T-1)} = (\omega_i^{(T)} - f_i^{(T)}) / \alpha$  $\omega_{i}^{(T-1)} = (\omega_{i}^{(T)} - \sigma_{i}^{(T)})/\alpha$ 

Rule 682(1010101010)

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$\alpha = 0.501$ DELA	AY	`₩₩₩ ₩₽₽₩ ₩₽₽₩	*060* -000- •000-	0, ;; ,0 11 · · 11 0 ^ = `0	<u>Nie</u> 215	₩,	()		

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$$0 \stackrel{\oplus}{\ominus} 0=0 1 \stackrel{\oplus}{\ominus} 0=0 1 \stackrel{\oplus}{\ominus} 1=0 0 \stackrel{\oplus}{\ominus} 1=1$$

# Reversible Parity Rule $({\sigma^{(0)}} = {\sigma^{(1)}})$

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α=0.501 ■₩[:] []]	0,00 %*** 0:3 40 * 000 40 **						11.5 X 11. 11. X 2 X					n <del>n</del> fe Feilen Feilen	α=0.501 ¤₩[:][]	000 400 0:5 40 000 40			)436) 1938) 7990)	2000 -2000 -2000	©, ≑,0 ⊪ · 4 0 * 0	31€ 215	214 214	**** -*** ****		())会会() (中))) ())(中)()	
α <b>=0.503</b> □₩[:][:][:]	0,00 %*** 0::0 40.0 0:0 40.0		1406) 0880 1974	010 Hiin 010		Ø						3711-18 (日本) (日本) (日本) (日本) (日本) (日本) (日本) (日本)	α=0.503 □₩[:][;]	000 XX 0:0 40 0:0 40 0:0 40					5,5,0 83 2'8'5	፝፞፞፝፝፞፞፞፞ቚዾ ፼፞፼ ዸቘዄ		1564631 ¥=¥ 4294934		n Heit Heit	
α <b>=0.505</b> ■₩[:][]]	0,0,0 %*** 0::0 40.0 000 ,999			Santa E = a Cauta						)间中间 (中间) (四中则)		8	α=0.505 □₩[]][]	0,00 %# 0,00 %# 0,00 %#	ŧ III	ikadi Gene 1979:		1000 1000 1000 1000 1000 1000 1000	2648. 1849)						
α=0.510 ■₩[:] []]		邇	<b>3</b> 33	懣	0-0 0-0	28	虃						α <b>=0.510</b> □ <b>≭</b> [:][]		. 494 . 406 . 406	<b>ii</b> ;	y : x - ≣ - λ : X					342 543	۲	2 # 1 #1 > # 2	8
α= <b>0.530</b> ■ <b>≭</b> [:] [']		æ	<b>88</b> )	5 0 3 603 203		œ			₩2# 12:22				α <b>=0.530</b> ■₩[:]	米 援	: 33	۲		澎	968 RA		4×4 ×0× **		畿	)	<b>1</b>
α=0.540 □ж[:][:]		æ	<b>)</b>	5.03 690 293								600) 800-33 1209	α <b>=0.540</b> ■₩[:][]	₩ #3	: :::	O	হায় হায়	癳	ама Хих амь	<b>33</b>		in the second			ॖॖॖॖॵख़ऄॖ ॾख़ऀख़ ऄॖॖॖॎक़ऻॖॖऄ
α= <b>0.600</b> ■₩[:] :::	* *	:::			8	505 505	68	<b>B</b>	濑	512 215	ено 2002 Сно		α=0.600 ·■₩[:][]	测计	: 🔛	**	æ	t t	ج	(c0.c) 1000 1000			×	246742 19 <b>88</b> 20 1912-19	25-25 2005 22-53
$\alpha = 0.700$ •• * * *	∷ ¥	33	83		ала Кор Түк	14.5 17.5 77.5	Ø			);;;;;; ;;;;;;;		5.5.7 1100111 120011	$\alpha = 0.700$ $\pi = 0.800$	96 <b>X</b>	×	::	[8]	æ	8	#:# 4:#	*		302		史》明 2 漢之 本 3 在
·•**	:: ·			(e)	ж	Ж	<u>(0)</u>	(e)	:::	:::	585 602 674	<b>.</b>	·= # · ::				(e)	(e)			(e)	(e)	:::	:::	(e)
$\alpha = 0.900$ $\alpha = 1.000$	:: •	=		(e)	۰	D		9	B	:::			$\alpha = 0.900$ $\alpha = 1.000$				<b>(6)</b>	(e)			2				<b>10</b>
•■₩₩				<b>36</b>			<b>(6)</b>	96)	D			<b>(</b> #)									•				

	Rey	versi Embeda	$\mathbf{ble}$	leg	gal	ECA wit	h $\alpha$ -1	mem Delav	ory	7	
Ahistoric	0.6	0.7	0.8	0.9	1.0	Ahistoric	0.6	0.7	0.8	0.9	1.0
Rule 18	<del>ó))(()(</del>	<del>()))))</del>	Ĵ	ŧ	Î	Rule 18	*	****	****	*	***
Rule 22		Â	Å	Î	Â	Rule 22	<b>\$</b>	₫ÛÛÛÛ		4III0III	
Rule 54	<del>tin</del> te	Â	Ĵ	Î	Î	Rule 50	<del>43808</del>	Î		<b>H</b>	<b>^</b>
Rule 00		& 8888 888	\$8888	38885	<b>\$888</b> 8	Rule 54			Â	Ŵ	
Rule 90					鲁	Rule 90	1	<b>\$</b> \$	<b>*</b> <b>*</b>	<b>♣</b> ♣	<b>*</b> <b>*</b>
Rule 94	Â		Â	Â		Rule 94				<b>1</b>	49993
Rule 122						Rule 122			*	<b>∦</b> #	*
	â					Rule 126					

```
function cam
SR=254;T=8;N=2*T+1;nat=6;plus=3;
alfa(1)=0.5;alfa(2)=0.6;alfa(3)=0.7;alfa(4)=0.8;alfa(5)=0.9;alfa(6)=1.0;
[srb]=binarynumber(SR);left=[N 1:N-1];right=[2:N 1];
for memo=1:2
for nal=1:nat; alpha=alfa(nal);
    [SIGMA,OMEGA,omega]=init(T,N,alpha);XX=SIGMA;
   switch memo
    case 1 % Embedded
    for t=1:T
      SIGMAH(t,:)=SIGMA;X_1=XX;XX=SIGMA;if(t==T)XSIGMA=SIGMA;end
      S=SIGMA; omega=(alpha*omega)+SIGMA; OMEGAX=OMEGA(t);
      for i=1:N
        if(2*omega(i)>OMEGAX)S(i)=1;end; % memory
        if(2*omega(i) < OMEGAX)S(i)=0;end
      end
      [SIGMA]=RULE(S,N,srb,left,right); % rule
      SIGMA=mod(SIGMA+X_1,2);
     end
     subplot(5+nat*plus,nat+plus,(nat+plus)*(memo-1)+nal);image(33*SIGMAH);axis('off');axis im
                          <sup>x</sup> reversion
     XX=SIGMA;SIGMA=XSIGMA;
     for t=1:T
      X_ 1=XX;SIGMAH(t,:)=SIGMA; XX=SIGMA;
      if(t > 1)
        S=SIGMA;OMEGAX=OMEGA(T-t+1);
        if(alpha>0);omega=(omega-X_1)/alpha;
        for i=1:N
                                         % back-memory
         if(2*omega(i)>DMEGAX)S(i)=1;end
         if(2*omega(i)<OMEGAX)S(i)=0;end
        end
      end
      end
      [SIGMA]=RULE(S,N,srb,left,right); % rule
      SIGMA=mod(SIGMA-X_1,2);
     subplot(5+nat*plus, nat+plus, (nat+plus)*(memo)+nal); image(33*SIGMAH); axis('off'); axis imag
     case 2 % delay
      [SIGMA,OMEGA,omega]=init(T,N,alpha);S=SIGMA;XX=SIGMA;
       for t=1:T
      SIGMAH(t,:)=SIGMA;X_1=XX;XX=SIGMAif(t==T)XSIGMA=SIGMA;end
      [S]=RULE(SIGMA,N,srb,left,right); % rule
      SIGMA=S;HS(t,:)=S;
      omega=(alpha*omega)+SIGMA;OMEGAX=OMEGA(t);
      for i=1:N
        if(2*omega(i)>OMEGAX)SIGMA(i)=1;end % memory
        if(2*omega(i) < OMEGAX)SIGMA(i)=0;end
      end
      SIGMA=mod(SIGMA+X 1.2);
     end
     subplot(5+nat*plus,nat+plus,(nat+plus)*(memo+2)+nal);image(33*SIGMAH);axis('off');axis image;
                          x reversion
     XX=SIGMA;SIGMA=XSIGMA;
     for t=1:T
      X_ 1=XX;SIGMAH(t,:)=SIGMA;XX=SIGMA;
      [S]=RULE(SIGMA,N,srb,left,right); % rule
      SIGMA=S;
      if(alpha>0)
        OMEGAX=OMEGA(T-t+1);
        for i=1:N
        if(2*omega(i)>OMEGAX)SIGMA(i)=1;end % back-memory
        if(2*omega(i)<OMEGAX)SIGMA(i)=0;end
         end
        omega=(omega-S)/alpha;
        end
      SIGMA=mod(SIGMA-X_1,2);
      end
     end
     subplot(5+nat*plus,nat+plus,(nat+plus)*(memo+2)+nal);image(33*SIGMAH);axis('off');ennis image;
end
end
print carevmemory.eps -depsc
function [SIGMA]=RULE(S,N,srb,left,right);
      for i=1:N
       SIGMA(i)=srb(8-(4*S(left(i))+2*S(i)+S(right(i))));
      end
function [SIGMA,OMEGA,omega]=init(T,N,alpha);
        SIGMA(1:N)=0; SIGMA((N+1)/2:(N+1)/2)=1;
        OMEGA(1)=1.0; omega(1:N)=0;
        for t=2:T;OMEGA(t)=1+alpha*OMEGA(t-1);end
function [BN] =binarynumber(rule);
        BN(1:8)=0; irtx=rule;
        for ix=1:8
        rest=mod(irtx,2);ratio=(irtx-rest)/2;BN(8-ix+1)=rest;irtx=ratio;
end
```

# $\sigma$ embedded s









# Reversible legal ECA with low $\alpha$ -memory $\alpha$ Delay





	$\operatorname{Re}_{\alpha}$	versi Embedd	$\mathbf{ble}_{\mathbf{ed}}$	leg	gal	ECA wit	$h \alpha - 1$	mem Delav	ory	7	
Ahistoric	<b>0.6</b>	0.7	<b>0.8</b>	0.9	1.0	Ahistoric	<u>0.6</u>	0.7	0.8	0.9	1.0
Rule 146		Â	Â	Î		Rule 146	<u>المجامعة</u>		Î	Î	ŝ
Rule 150				<b>8</b> 00	<b>8</b> 20 20	Rule 150			Å	Å	, Ņ
Rule 178	4 8888 88	Â		ÂCCC	() 00000	Rule 178		<b>~#</b> 88888 88		<b>HERE</b> KKY	
Rule 102		& 8888 8888	<b>\$8888</b>	38888	<b>\$888</b> 8	Rule 182		Â.	Â	Ŵ	
	Å		<b>4</b> 00000	<b>\$</b> 00000		Rule 218					
		Â	Å	*	*	Rule 222			â		
Rule 250	8		<b>\$</b>	ô	*	Rule 250			â	*	
Rule 254	â			â	â	Rule 254					

#### 1D r=2 CA





Rule III block CA with  $\tau = 3$  majority memory, starting at random.

## Two-dimensional block cellular automata





The HPP block CA rule with delay memory and the Density Classification Task



Percentage of correctly classified densities and average time up to convergence. 2D lattices of size  $n \times n$ . 10<sup>5</sup> binomially generated ICs.

	$\tau = 3$	$\tau = 4$	$\tau = 5$	$\tau = 6$	$\tau = 7$	$\alpha = 0.55$	$\alpha = 0.60$	$\alpha = 0.65$	$\alpha = 0.70$	$\alpha = 0.75$
n=22 n=21	88.732 42 82.762 87	92.840 41 87.331 87	$\begin{array}{ccc} 87.467 & 51 \\ 81.892 & 105 \end{array}$	89.804 46 84.152 100	$\begin{array}{rrrr} 86.715 & 61 \\ 81.472 & 127 \end{array}$	92.988 43 <b>88.601 99</b>	<b>94.212 40</b> 86.771 92	$\begin{array}{rrrr} 92.109 & 39 \\ 85.951 & 95 \end{array}$	$\begin{array}{rrrr} 90.786 & 43 \\ 84.892 & 97 \end{array}$	90.552 $48$ 84.512 114
n=32 n=31	$\begin{array}{ccc} 87.297 & 65 \\ 81.782 & 136 \end{array}$	$\begin{array}{rrrr} 91.362 & 62 \\ 84.712 & 136 \end{array}$	85.853 78 80.682 164	87.840 73 82.972 157	84.417 93 80.272 195	$\begin{array}{rrrr} 92.378 & 59 \\ 85.770 & 149 \end{array}$	92.159 61 84.588 141	$\begin{array}{rrr} 90.319 & 61 \\ 84.357 & 150 \end{array}$	88.752 68 83.241 152	88.331 77 82.815 182
n=42 n=41	86.101 88 80.272 186	89.818 85 83.552 186	84.426 105 79.432 220	86.449 100 82.082 213	83.665 126 78.472 248	91.442 80 83.792 200	$\begin{array}{rrr} 90.610 & 85 \\ 83.392 & 191 \end{array}$	88.836 86 83.202 206	86.916 96 82.402 208	86.813 107 81.682 244

Alonso-Sanz, R. (2013). Cellular automaton with memory and the density classification task. J. of Cellular Automata (in press)

The Rule III block CA rule with delay memory and the DCT



Evolution of density with low delay memory. Fifty simulations.

Percentage of correctly classified densities and average time up to convergence in n-size registers.  $10^5$  binomially generated ICs.

Alonso-Sanz, R. (2013). Cellular automaton with memory and the density classification task. J. of Cellular Automata (in press)

# SALVADOR DALI



The Persistence of Memory



Disintegration of the Persistence of Memory