Computational complexity of majority automata under different updating schemes

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Automata network

An *Automata Network* is a triple $\mathcal{A} = (G, Q, f_i : i \in V)$, where

- $G = (V, E)$ is a simple undirected graph and $V = \{1, \ldots, n\}$.
- Q the set of states ($Q = \{0, 1\}$)
- $f_i : \{0, 1\}^n \to \{0, 1\}$ is the transition function associated to the vertex i.

We say that vertices in state 1 are *active* while vertices in state 0 are *passive*.

Updating Schemes

An *updating scheme* (US) of the automaton $\mathcal{A}$ is a function

$$\phi : V \to \{1 \ldots |V|\}$$

st. if u and v are vertices and $\phi(u) < \phi(v)$ then the state of u is updated before v, and if $\phi(u) = \phi(v)$ then nodes u and v are update at the same time.

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- **Block sequential:**

$$V = \cup_{i=1}^{k} V_i, \qquad \cap_{i=1}^{k} V_i = \emptyset, \qquad \phi|_{V_i} = i$$

  The vertex set is partitioned into several subsets, st. into the same set every vertex is updated at the same time, and different subsets are updated sequentially in some order.

Trajectory of a configuration

Let $x \in \{0,1\}^n$ be a configuration of an automaton. The trajectory $T^{\phi}(x)$ of x with the updating scheme ϕ is the set

$$T^{\phi}(x) = \{x(t) : t \geq 0\}$$

where $x(0) = x$ and $x(t+1)$ is obtained from $x(t)$ after every vertex is updated according to ϕ.

The trajectory of x *enters in a limit cycle of period* p if $|T(x(t))| = p$ for some $t \geq 0$.
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$\tau_\phi(\mathcal{A}) = \max\{\tau_\phi(x) : x \in \{0,1\}^n\}$ is the transient length of $\mathcal{A}$.

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One Cell Prediction: OCP

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- and $v \in V$ a vertex initially passive ($x_v = 0$),

Does there exists $y \in T^{\phi}(x)$ such that $y_v = 1$?

Majority automata

Here we will consider only *majority functions*, i.e.:

$$f_i(x) = \begin{cases} 1 & \text{if} \quad \sum_{j \in N(i)} x_i > \frac{|N(i)|}{2} \\ 0 & \text{if} \quad \sum_{j \in N(i)} x_i \leq \frac{|N(i)|}{2} \end{cases}$$

where $N(i)$ is the set of neighbors of vertex i.

An automata network with this rule is called a *majority automata*.

Parallel and sequential US.

Theorem

For parallel and sequential updating schemes, **OCP** *is in* **P**

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$|T(x)| = |\{x(t) \ : \ t > 0\}|$ is $poly(n)$?

[E. Goles, F. Fogelman, D. Pellegrin]

If $\{f_1, \dots, f_n\}$ are threshold functions with weights matrix A and threshold vector b.

$$E_{syn}[x(t)] = -\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i(t-1) x_i(t) + \sum_{i=1}^{n} b_i (x_i(t) + x_i(t-1))$$

$$E_{seq}[x(t)] = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i(t) x_j(t) + \sum_{i \in V} b_i x_i(t)$$

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  Sequential US $\rightarrow$ reach only fixed points.

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  Sequential US $\rightarrow$ reach only fixed points.
- $\tau(\mathcal{A})$ is $\mathcal{O}(n^3)$

Block sequential US

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Theorem

There is a block sequential update scheme in a majority automata, such that each block has cardinality 2 and the limit cycle has a super-polynomial length.

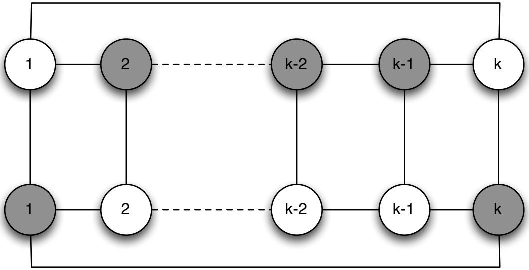
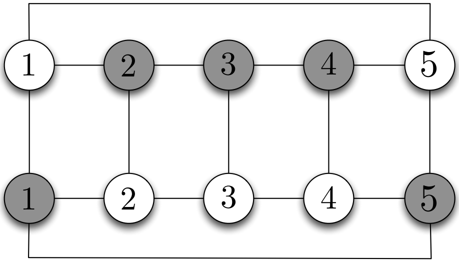
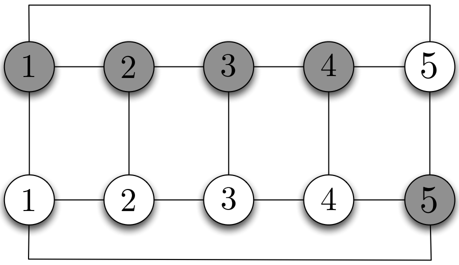


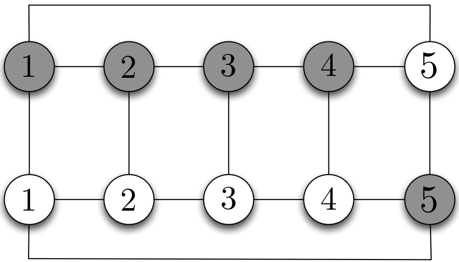
Figure: Ladder



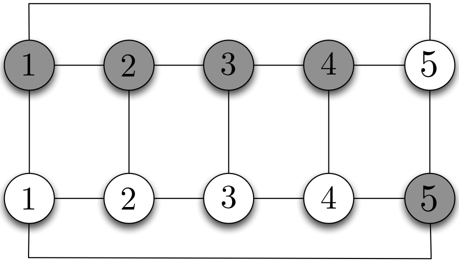
$t = 0$



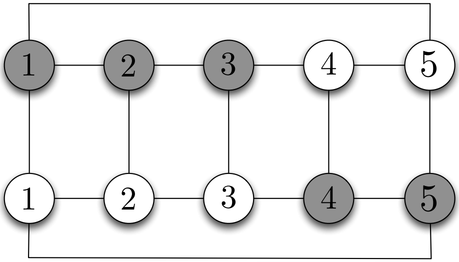
$t = 0 - 1$



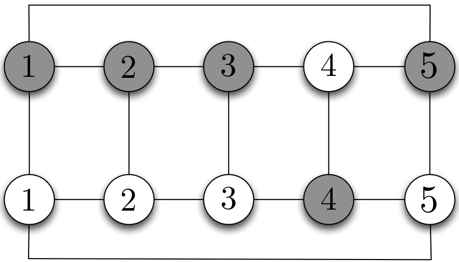
$t = 0 - 2$



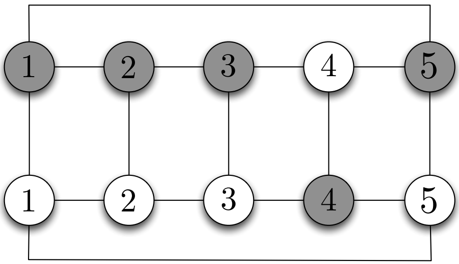
$$t = 0 - 3$$



$t = 0 - 4$



$$t = 0 - 5 = 1$$



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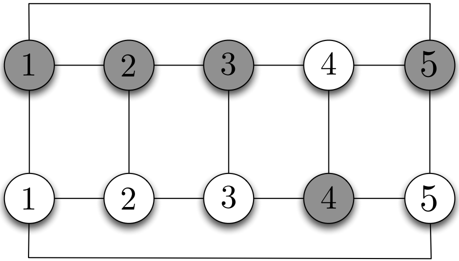

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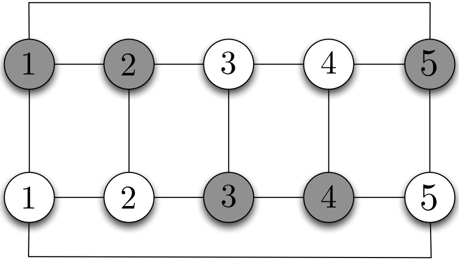

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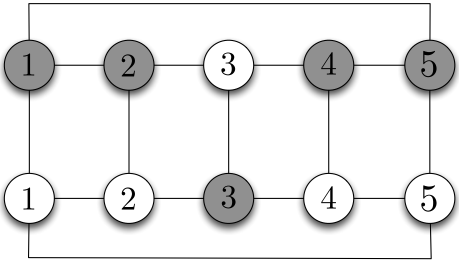
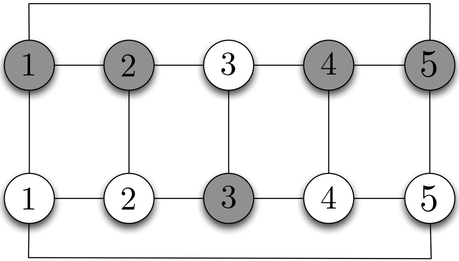
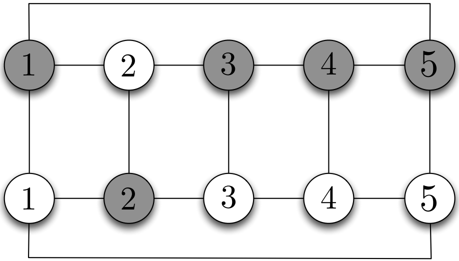


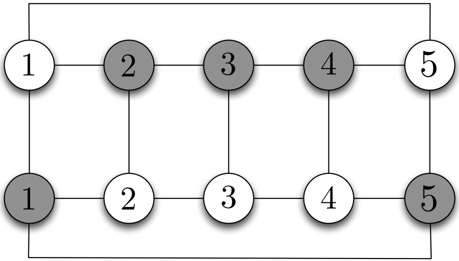
$t = 1 - 4$



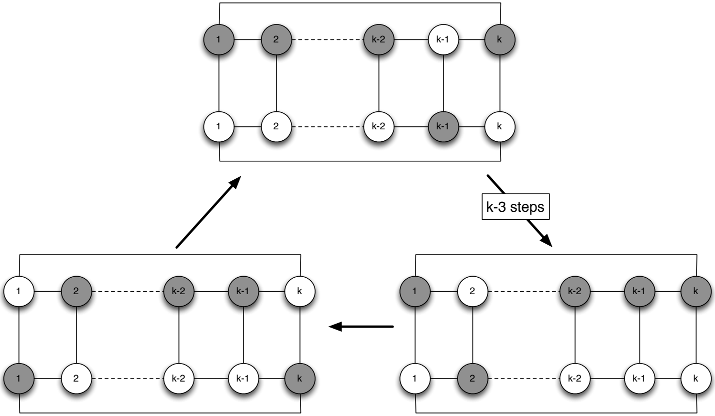
$$t = 1 - 5 = 2$$

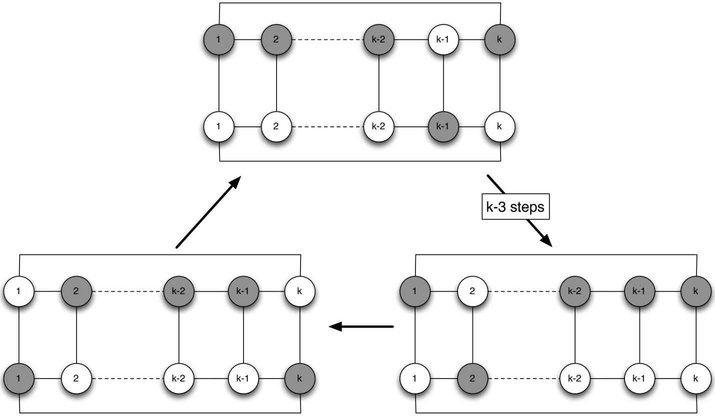


$t = 3$

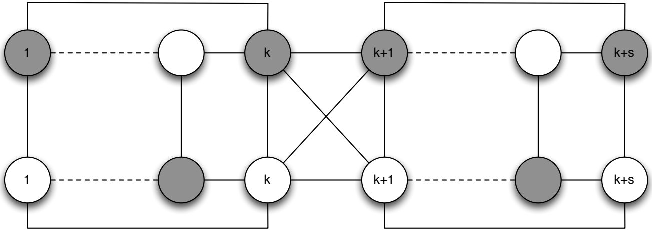


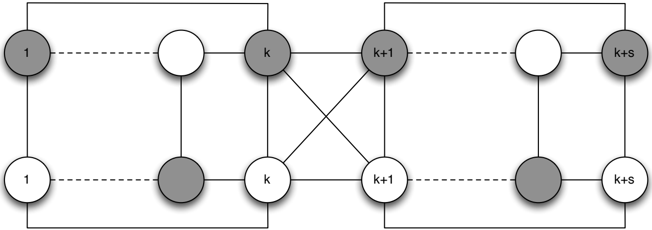
$t = 4$


1
2
k-2
k-1
k
k-3 steps



Limit cycle of length $k - 1$


1
k
k+1
k+s
1
k
k+1
k+s



Limit cycle of length $lcm(k-1, s-1)$

Let m be a positive integer, and let $\pi(m)$ the number of primes not exceeding m.

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Let G the graph obtained from $\pi(m)$ ladders of sizes $(p_1 + 1), (p_2 + 1), \ldots, (p_{\pi(m)} + 1)$, where $\{p_1, p_2 \ldots, p_{\pi(m)}\}$ the first $\pi(m)$ primes.

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Then

$$V(G) \leq \sum_{i=1}^{\pi(m)} 2(p_i + 1) \leq 2\pi(m)(m+1)$$

$$\text{limit cycle of } G = lcm(p_1, \ldots, p_{\pi(m)}) = \prod_{i=1}^{\pi(m)} p_i = e^{\theta(m)}$$

where $\theta(m) = \sum_{i=1}^{\pi(m)} \log(pi)$.

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From the Prime Number Theorem:

$$lcm(p_1, \ldots, p_{\pi(m)}) \geq e^{\Omega(\sqrt{|V(G)| \log(|V(G)|}))}$$

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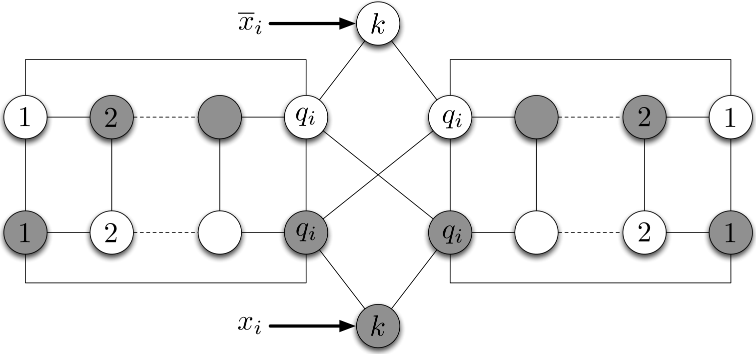
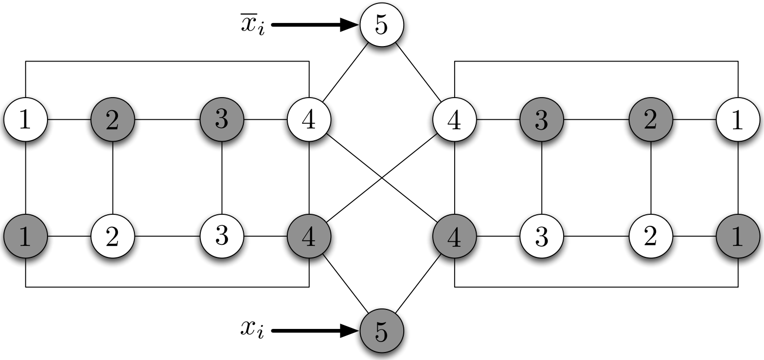
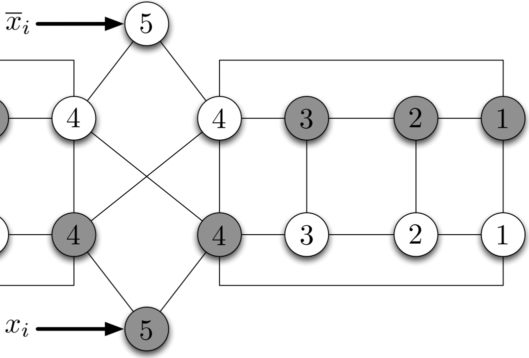
Proof: Reduce $3 - SAT$.



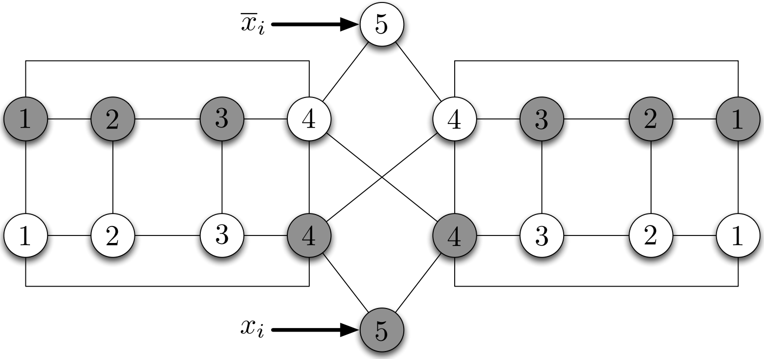
Gadget for variable x_i (positive and negative literals)



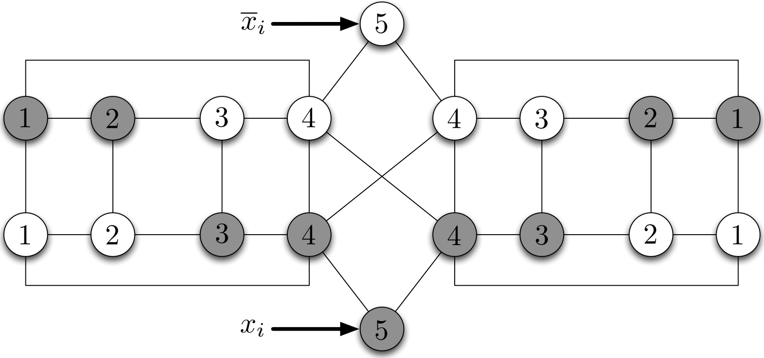
$t = 0$



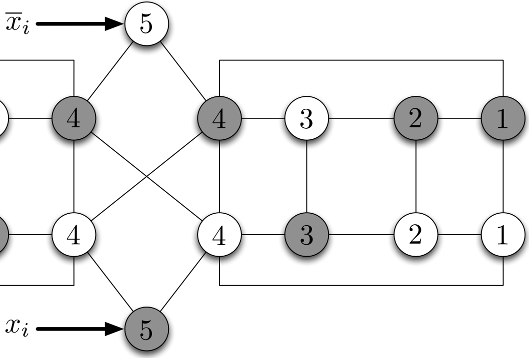
$t = 0 - 1$



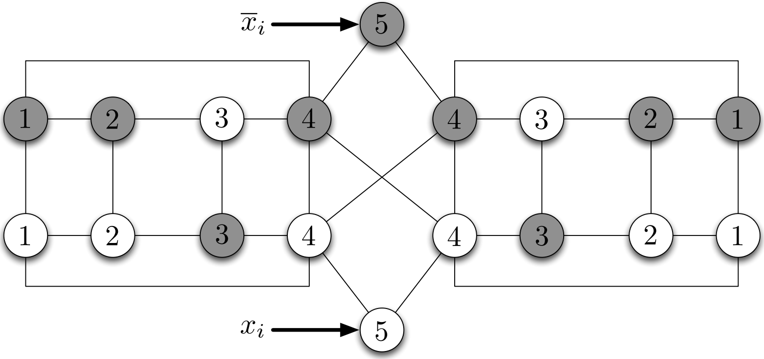
$t = 0 - 2$



$t = 0 - 3$



$t = 0 - 4$



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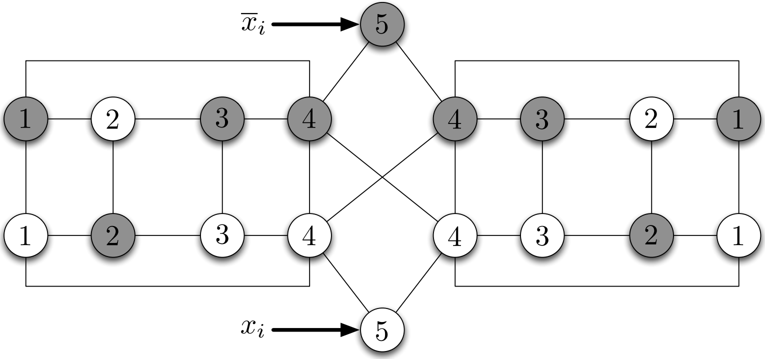
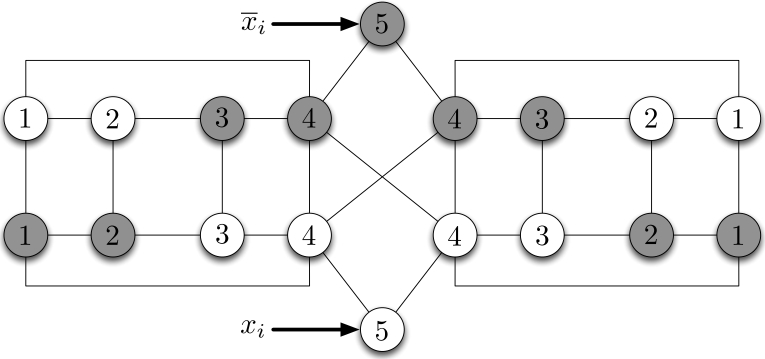
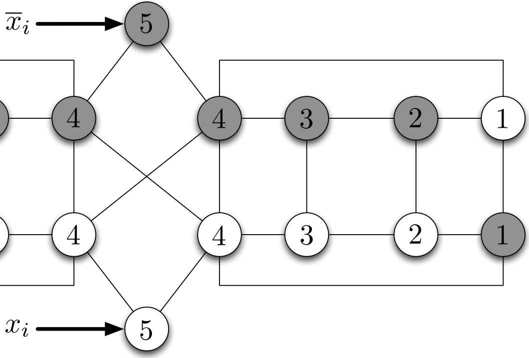


$t = 2$



$t = 2 - 1$



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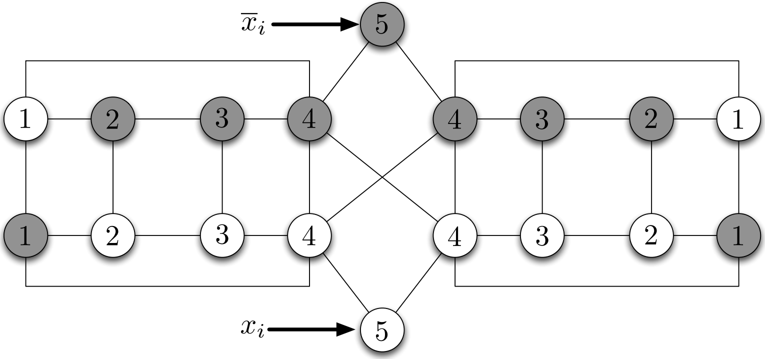
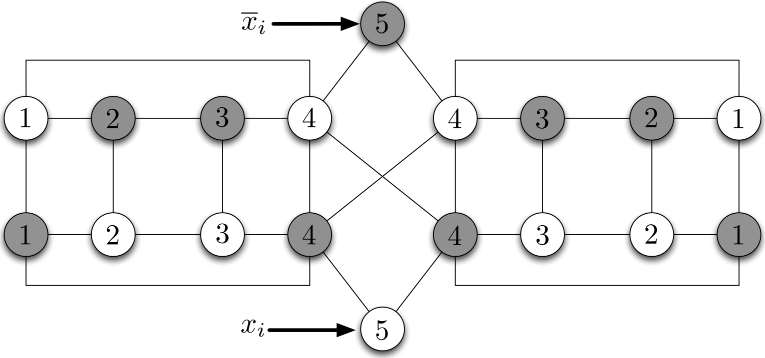


$t = 2 - 3$



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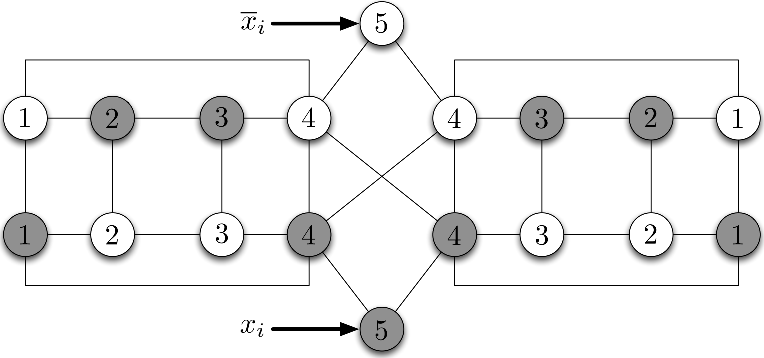
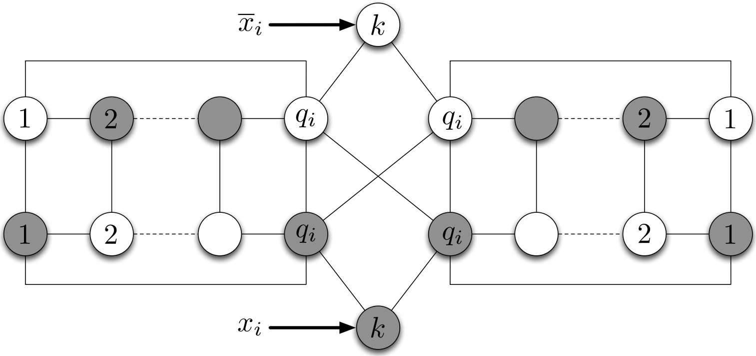
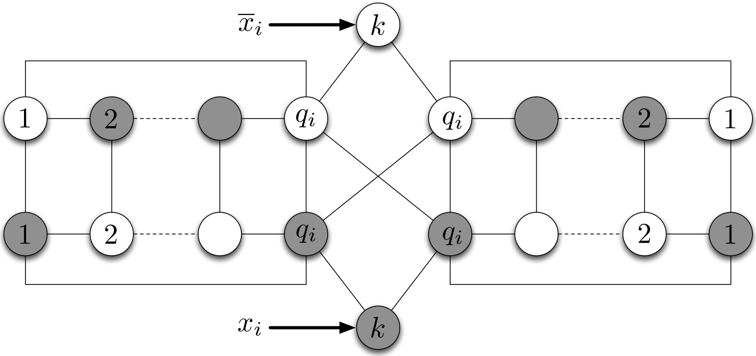


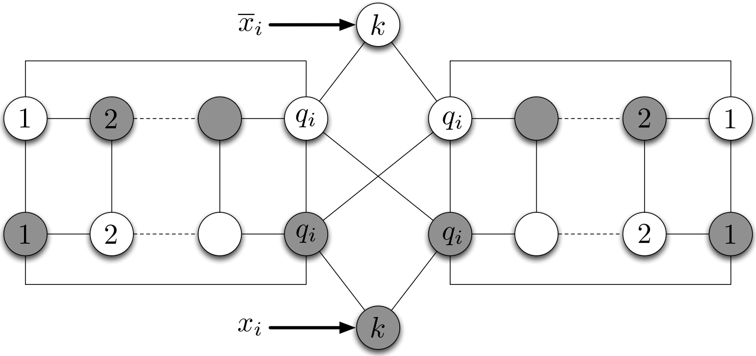
$t = 2 - 5 = 3$



$\overline{x}_i$
k
1
2
q_i
q_i
2
1
1
2
q_i
q_i
2
1
x_i
k



$q_i = p_i + 1$ where p_i i-th prime



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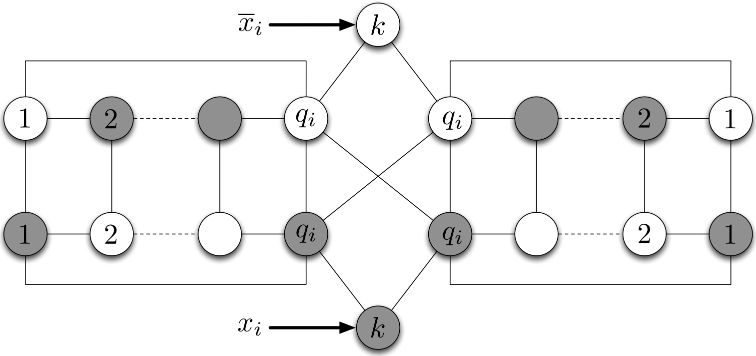

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$k = p_n + 2$

$x_i = 1$ in steps multiple of p_i, and $x_i = 0$ otherwise

Then combination of values of variables $(x_1, x_2, x_3, \ldots, x_n)$, $x_i \in \{0, 1\}$ happens in step

$$p_1^{x_1} \times p_2^{x_2} \times \cdots \times p_n^{x_n}$$

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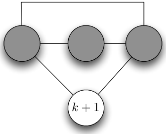
Any possible combination of input values can be simulated.

$$C_i = (x_{i_1} \vee \overline{x}_{i_2} \vee \overline{x}_{i_3})$$



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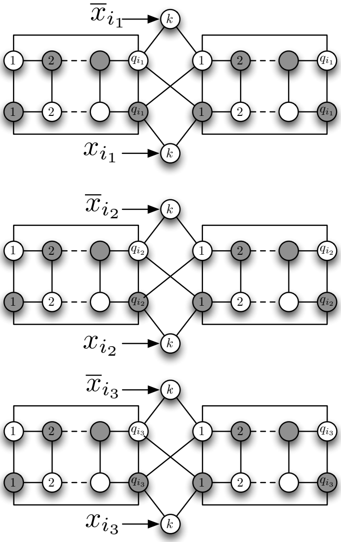

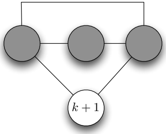

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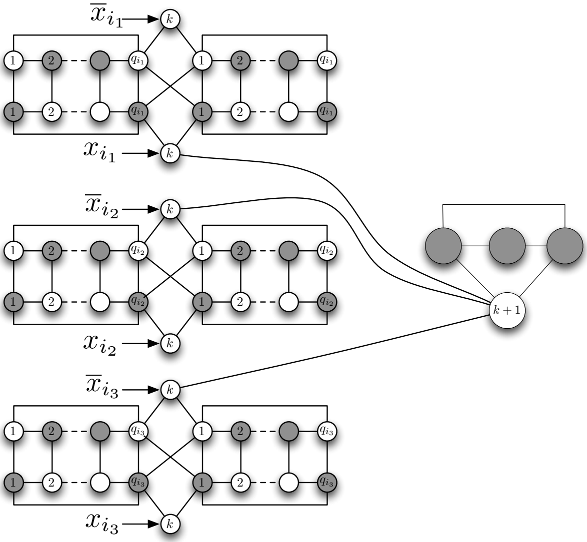
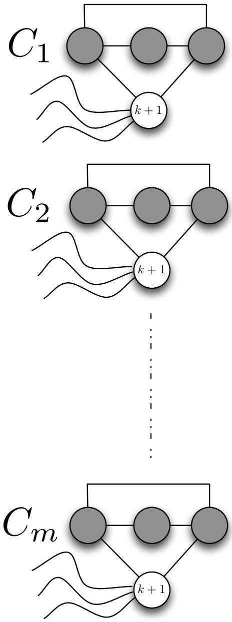
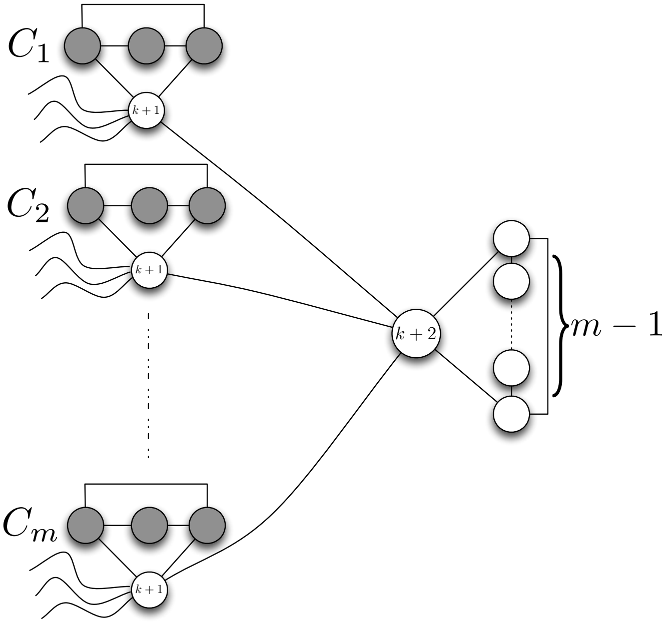




C_1
$k+1$
C_2
$k+1$
C_m
$k+1$



C_1
$k+1$
C_2
$k+1$
C_m
$k+1$
$k+2$
$m-1$

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Danke!