Color Blind Cellular Automata

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Cellular automata and color blindness

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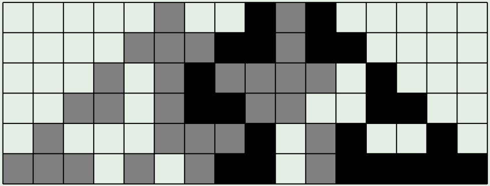
- Color blind cellular automata are the ones that can be defined without referring to any individual colors.
- Also: a cellular automaton is *captive* if its local rule always outputs one of the input colors.

Example

The radius-1 cellular automaton f on $\{0,1,2\}^{\mathbb{Z}}$ defined by

$$f_{\mathrm{loc}}(a,b,c) = \begin{cases} c, & \text{if } a = b \neq c, \\ a, & \text{if } a \neq b = c, \\ b, & \text{otherwise} \end{cases}$$

is clearly color blind. It always chooses the symbol in its neighborhood that is in the minority, or acts as the identity CA if such a symbol does not exist.



Elementary examples of color blind CA

Table: The even-numbered color blind elementary CA. The variables v_1, v_2 and v_3 denote inputs to the local rule, and v_4 is its output.

CA	Color blind equation	Description
142	$(v_4 \neq v_2) \iff (v_1 = v_2 \neq v_3)$	?
150	$(v_1 = v_2) \iff (v_4 = v_3)$	Sum of neighborhood
170	$v_4 = v_3$	Left shift
178	$(v_4 = v_2) \iff (v_1 = v_2 = v_3)$	Flip unless all equal
204	$v_4 = v_2$	Identity
212	$(v_4 \neq v_2) \iff (v_1 \neq v_2 = v_3)$	Mirrored 142
232	$(v_4 \neq v_2) \iff (v_1 \neq v_2 \neq v_3)$	Majority
240	$v_4 = v_1$	Right shift

Color blindness vs. captivity

Lemma

Let $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ be a color blind CA. Then $f_{\mathrm{loc}}(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ whenever $|\{a_1, \ldots, a_n\}| < |S| - 1$.

Proof.

Suppose that we have $|\{a_1, \ldots, a_n\}| < |S| - 1$, but $a = f_{\mathrm{loc}}(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$. Then, there exists

$$b \in S \setminus \{a, a_1, \ldots, a_n\}.$$

Now, f does not commute with the transposition $(a\ b)$. $\square$

Color blind equations

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- A set of words $W \subset S^n$ *defines a function* if for each $w \in S^{n-1}$, there is a unique extension $wa \in W$. The function maps $w \mapsto a$.
- We say E is *captive on* S if the last letter of w occurs at least twice in w for all $w \in E(S)$, and *captive*, if it is captive on S for all finite S.

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- If $n = 2r + 2$ and $E(S)$ defines a function from S^{2r+1} to S, we let $f_E^S : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ be the cellular automaton whose local function it is. We say f_E^S is *defined by a color blind equation*.

A characterization of color blind CA

Lemma

A set of words $W \subset S^n$ is defined by a color blind equation if and only if it is closed under symbol permutations.

Corollary

A CA $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is (captive and) color blind if and only if it is defined by a (captive and) color blind equation.

Side note: Color blind CA running on infinite alphabets

- Suppose E over variables $\{v_1, \dots, v_{2r+2}\}$ is a captive color blind equation and defines a function from S^{2r+1} to S for all S
- in practise, this means E always equates v_{2r+2} with one of the v_i, and takes into account all possible partitions.

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- in practise, this means E always equates v_{2r+2} with one of the v_i, and takes into account all possible partitions.
- Then, the captive color blind CA f_E^S is defined for $S = [0, n]$ for all $n \in \mathbb{N}$.
- Further, if $m \leq n$, then $f_E^{[0,n]}|_{[0,m]^{\mathbb{Z}}} = f_E^{[0,m]}$.
- We can take a 'limit' of this, obtaining a 'cellular automaton' f_E on $\mathbb{N}^{\mathbb{Z}}$!

An intrinsically universal cellular automaton

- By a suitable coding, we can easily make a color blind cellular automaton simulate any other cellular automaton.

Theorem

For any alphabet S with $|S| \geq 2$, there exists an intrinsically universal captive color blind cellular automaton on $S^{\mathbb{Z}}$.

- Thus, color blind cellular automata can, in a sense, be as complicated as any general cellular automata.

On the number of color blind cellular automata

Definition

Let $\mathcal{C}$ be a family of cellular automata on $S^{\mathbb{Z}}$. The *density of* $\mathcal{C}$ is defined as

$$d(\mathcal{C}) = \limsup_{n \to \infty} \frac{1}{n} \log_{|S|} \log_{|S|} |\mathcal{C}_n|, \qquad (1)$$

where $\mathcal{C}_n$ denotes the set of cellular automata in $\mathcal{C}$ that can be defined on the neighborhood $[-\lfloor n/2 \rfloor, \lceil n/2 \rceil]$.

Color blind cellular automata are abundant in the sense of the previous definition. Note that the set $\mathcal{CA}$ of all cellular automata on $S^{\mathbb{Z}}$ has density 1.

Proposition

Denote by $\mathcal{CB}$ the set of captive color blind cellular automata on $S^{\mathbb{Z}}$. Then $d(\mathcal{CB}) = 1$.

Typhlotic cellular automata

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- We have seen that color blindness, that is, commuting with symbol permutations, is not very restrictive.
- What about commuting with all symbol *maps*?
- If a cellular automaton $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ commutes with all symbol maps, we say it is *typhlotic*.

Characterization of typhlotic CA

- Interestingly, there are almost no typhlotic CA if $|S| > 2$, but there are many of them if $|S| = 2$.

Theorem

If $|S| \geq 3$, the typhlotic CA $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ are exactly the shift maps. If $|S| = 2$, they are exactly the captive color blind CA.

Proof idea in the case $|S| = 2$

- If $|S| = 2$, then the only symbol maps are the symbol permutations and the constant maps. Commuting with the first is equivalent to color blindness, and the latter is equivalent to captivity.

Proof idea in the case $|S| \geq 3$

- For any S, we can prove that there exists a set of subsets Q of the neighborhood of the cellular automaton such that $f_{\mathrm{loc}}(a_1, \ldots, a_n) = b$ if and only if $\{i \mid a_i = b\} \in Q$.

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- Then, necessarily, Q satisfies the assumptions of the following lemma, and is thus an ultrafilter:

Lemma

Let X be a nonempty set, let $k \in \mathbb{N}$ with $k \geq 3$, and let $Q \subset 2^X$ have the property that for all partitions $(A_1, \ldots, A_k)$ of X, exactly one A_i is in Q. Then Q is an ultrafilter. Furthermore, every ultrafilter satisfies the property for every $k \geq 1$.

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- But an ultrafilter on a finite set is of the form $\{A \mid i \in A\}$ for some i, so f is a shift map!

Color blind group homomorphic cellular automata

- We have seen that color blindness, that is, commuting with symbol permutations, is not very restrictive.
- On the other hand, typhloticness has been shown to be restrictive for large alphabets, but non-restrictive for small ones.

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- We have seen that color blindness, that is, commuting with symbol permutations, is not very restrictive.
- On the other hand, typhloticness has been shown to be restrictive for large alphabets, but non-restrictive for small ones.
- We next consider cellular automata which are color blind, and group homomorphisms for some group $G^{\mathbb{Z}}$.
- Similarly to the case of typhlotic cellular automata, it turns out that if $|G|$ is small, then the automaton can be complicated, but if $|G|$ is large, the cellular automaton is a shift map.

Characterization

Theorem

Let G be a finite group, and let $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be a homomorphic cellular automaton. Then, f is color blind iff one of the following (partially overlapping) conditions holds.

- $G = \mathbb{Z}_2$, $G = \mathbb{Z}_2^2$ *or* $G = \mathbb{Z}_3$, *and* f *fixes unary points,*
- $G = \mathbb{Z}_2$ *or* $G = \mathbb{Z}_2^2$, *and* f *is a sum of an odd number of distinct shifts,*
- $G = \mathbb{Z}_3$, *and* f *is a sum of* $3k+1$ *shifts for some* k,
- $|G| > 4$ *or* $G = \mathbb{Z}_4$, *and* f *is a shift map.*

Thank you for listening!