I F I G
Research
Report

Institut für Informatik

# Comments on Monoids Induced by NFAs

Markus Holzer

IFIG Research Report 2301
March 2023

Institut für Informatik
JLU Gießen
Arndtstraße 2
35392 Giessen, Germany
Tel: +49-641-99-32141
Fax: +49-641-99-32149
mail@informatik.uni-giessen.de
www.informatik.uni-giessen.de

IFIG Research Report
IFIG Research Report 2301, March 2023

# Comments on Monoids Induced by NFAs

Markus Holzer[1]

Institut für Informatik, Universität Giessen
Arndtstr. 2, 35392 Giessen, Germany

**Abstract.** We summarize known results on the transformation monoid of nondeterministic finite automata (NFAs) from semigroup theory. In particular, we list what is known from the literature on the size of monoids induced by NFAs and their (minimal) number of generators—a comprehensive list of these generators is given in the Appendix. It is shown that any language accepted by an $n$-state NFA has a syntactic monoid of size at most $2^{n^2}$. This bound is reachable by the generators of the semigroup $B_n$ of $n \times n$ Boolean matrices with the usual matrix multiplication except that we assume $1 + 1 = 1$. The number of these generators grows exponentially in $n$. This is a significant difference to the deterministic case, where three generators suffice to generate all elements of $T_n$. Moreover, we prove a lower bound for the NFA-to-DFA conversion using Lambert's $W$ function.

MSC Classification:
20M20: Semigroups of transformations, etc.
20M35: Semigroups in automata theory, linguistics, etc.
68Q45: Formal languages and automata
68Q70: Algebraic theory of languages and automata

Additional Key Words and Phrases: Nondeterministic finite automata, transformation monoid, semigroups, $n \times n$ Boolean matrices, semigroup theory, lower bound, NFA-toDFA conversion, Lambert $W$ function.

[1] Email: holzer@informatik.uni-giessen.de

## 1 Introduction

We assume the reader to be familiar with the notion of finite automata, in particular, with deterministic finite automata (DFAs). Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA with $\delta : Q \times \Sigma \to Q$. Then every letter $a \in \Sigma$ defines a total function $a : Q \to Q$ in the natural way, that is, $a(p) = q$ if $\delta(p, a) = q$. The functions $a$, for $a \in \Sigma$, are the generators of a *transformation monoid* that is a subset of $T_n$ with $n = |Q|$. Obviously, the size of $T_n$ is upper bounded by $n^n$. The generators of the transformation monoid $T_n$ are nicely characterizable. The *kernel* of a transformation $\alpha$ is the equivalence relation $\equiv$, which is induced by $i \equiv j$ if and only if $(i)\alpha = (j)\alpha$ (function application is to the left). Then by Salomaa [15] the following result was shown, which was re-discovered several times during the years; for instance see Dénes [2].

**Theorem 1 (Salomaa).** *Assume $n \geq 3$. Then three elements of $T_n$ generate all transformations of $T_n$ if and only if two of them generate the symmetric group $S_n$ and the third has kernel size $n - 1$. Moreover, no less than three elements generate all transformations from $T_n$.*

This gives us the result, that the largest subsemigroup of $T_n$ generated by three or more elements has full size $n^n$. The size of the largest subsemigroup for two generators was studied in [6], where a lower bound of $n^n(1 - \frac{2}{\sqrt{n}})$ and a trivial upper bound of $n^n - n! + g(n)$ was obtained—here $g(n)$ refers to Landau's function on the largest order of permutations on $n$ elements. A slightly better lower bound of $n^n(1 - \frac{4}{n})$ for odd $n \geq 70$ was presented in [12].

The syntactic monoid for a given language $L \subseteq \Sigma^*$, is defined by the *syntactic congruence* $\sim_L$ over $\Sigma^*$ where $v_1 \sim_L v_2$ if and only if $uv_1w \in L \iff uv_2w \in L$ for every $u, w \in \Sigma^*$. Then the *syntactic monoid* is the quotient monoid $M(L) = \Sigma^*/\sim_L$, where the concatenation of equivalence classes $[u]_{\sim_L} \cdot [v]_{\sim_L} = [uv]_{\sim_L}$ serves as the monoid operation. The syntactic monoid of a regular language $L$ is the smallest monoid recognizing the language under consideration (with respect to the division relation) and it is isomorphic to the transformation monoid of the minimal deterministic finite automaton accepting $L$. Here a language $L \subseteq \Sigma^*$ is *recognizable* if and only if there exists a finite monoid $M$, a morphism $\varphi : \Sigma^* \to M$, and a subset $N \subseteq M$ such that $L = \varphi^{-1}(N)$, which in turn is equivalent to the regularity (acceptance by a finite state machine) of $L$.

## 2 Transformation Monoid of a NFA

Next let us define a transformation monoid for nondeterministic finite automata (NFA). To this end we use the monoid of binary relations on a set of size $n$—this algebraic structures is isomorphic to the semigroup $B_n$ of $n \times n$ Boolean matrices with the usual matrix multiplication except that we assume $1 + 1 = 1$. Any binary relation $R$ on the $n$-set $\{1, 2, \ldots, n\}$ can be represented naturally by a $n \times n$ Boolean matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if and only if $i\,R\,j$. We closely follow the lines of [13] and [14]. Let an NFA $A = (Q, \Sigma, \delta, q_0, F)$ be given. Here we assume that $\delta : Q \times \Sigma \to 2^Q$. Then for every letter $a \in \Sigma$ define the relation $R_a \subseteq Q \times Q$ as follows: $(p, q) \in R_a$ if and only if $q \in \delta(p, a)$. As operation on the relations we use composition of relations. Then the morphism $h : \Sigma^* \to Q \times Q$ defined *via* $h(a) = R_a$, for every $a \in \Sigma$, recognizes the language $L(A)$, i. e., we can write $L(A) = h^{-1}(S)$, where $S = \{\, R \subseteq Q \times Q \mid (q_0, q) \in R, \text{ for some } q \in F \,\}$. The size of the relation monoid is upper bounded by $2^{n^2}$ with $n = |Q|$.

Let me give an example. Consider the NFA $A = (Q, \{a, b\}, \delta, q_0, F)$ with state set $Q = \{1, 2\}$, initial state $q_0 = 1$, set of final states $F = \{2\}$, and $\delta(1, a) = \{1, 2\}$, $\delta(1, b) = \{2\}$, and finally $\delta(2, a) = \delta(2, b) = \{2\}$. The automaton $A$ is depcited in the left of Figure 1. On the right


**Fig. 1.** The NFA $A$ (left) and the minimal DFA (right) accepting the language $L(A)$. The transition monoid of $A$ has size 3, while the transition monoid, which is equivalent to the syntactic monoid of $L(A)$, contains only 2 elements.

of the figure the minimal DFA accepting $L(A)$ is shown. The transformation monoid induced by $A$ contains the three elements *id*, $a$, and $b$ satisfying $a^2 = a$ and $ab = ba = b^2 = b$. These elements represent the matrices

$$id = \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix}, \quad a = \begin{pmatrix} 1\,1 \\ 0\,1 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 0\,1 \\ 0\,1 \end{pmatrix}.$$

On the other hand, the transformation monoid of the corresponding minimal DFA accepting $L(A)$, which is isomorphic to the syntactic monoid of $L(A)$, has only two elements, namely *id* and $a$ with the equations $a = b$ and $a^2 = ab = ba = b^2 = a$.

Before we continue we introduce some further notations on semigroup theory. Let $S$ be a semigroup. For $a$ and $b$ in $S$, we say that $a \,\mathcal{L}\, b$ if they generate the same *left ideal*, i. e., $S^1a = S^1b$, where $S^1$ is $S$ with an identity adjoined, and $a \,\mathcal{R}\, b$ if they generate the same *right ideal*, that is, $aS^1 = bS^1$. We also define the *two-sided ideal* $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$, the smallest equivalence relation containing both $\mathcal{L}$ and $\mathcal{R}$. These equivalence relations are well-known and called *Greens relations*. The relations $\mathcal{L}$ and $\mathcal{R}$ commute, and $\mathcal{D} = \mathcal{D} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. An element $a$ of a semigroup $S$ is *regular* if there exists an element $b$ in $S$ such that $aba = a$. Moreover, a semigroup $S$ where every element is regular is said to *regular*. Finally, we introduce notations particular for $B_n$. A Boolean matrix $\alpha$ in $B_n$ is called *prime* if $\alpha$ is *not* in $S_n$ and whenever $\alpha = \beta\gamma$, where both elements $\beta$ and $\gamma$ are in $B_n$, then either $\beta$ or $\gamma$ is in $S_n$. Finally, two matrices $\alpha$ and $\beta$ in $B_n$ are *similar* if $\alpha$ can be obtained by row and/or column permutation from $\beta$. Now we are ready to continue our journey on generators for $B_n$.

Concerning the number of generators the situation is more involved compared to the symmetric group $S_n$, where two generators suffice, and that of $T_n$, with three ones. In [9] it is proven that the semigroup $B_n^r$ generated by the regular elements of $B_n$ has a generating set consisting of four elements.[1] These are the following four elements

$$\begin{pmatrix} 0\,1\,0\,0\,\ldots\,0\,0 \\ 0\,0\,1\,0\,\ldots\,0\,0 \\ 0\,0\,0\,1\,\ldots\,0\,0 \\ \vdots\,\vdots\,\vdots\,\vdots\,\ddots\,\vdots\,\vdots \\ 0\,0\,0\,0\,\ldots\,0\,1 \\ 1\,0\,0\,0\,\ldots\,0\,0 \end{pmatrix}, \quad \begin{pmatrix} 0\,1\,0\,\ldots\,0\,0 \\ 1\,0\,0\,\ldots\,0\,0 \\ 0\,0\,1\,\ldots\,0\,0 \\ \vdots\,\vdots\,\vdots\,\ddots\,\vdots\,\vdots \\ 0\,0\,0\,\ldots\,1\,0 \\ 0\,0\,0\,\ldots\,0\,1 \end{pmatrix}, \quad \begin{pmatrix} 1\,0\,0\,\ldots\,0\,0 \\ 1\,1\,0\,\ldots\,0\,0 \\ 0\,0\,1\,\ldots\,0\,0 \\ \vdots\,\vdots\,\vdots\,\ddots\,\vdots\,\vdots \\ 0\,0\,0\,\ldots\,1\,0 \\ 0\,0\,0\,\ldots\,0\,1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1\,0\,0\,\ldots\,0\,0 \\ 0\,1\,0\,\ldots\,0\,0 \\ 0\,0\,1\,\ldots\,0\,0 \\ \vdots\,\vdots\,\vdots\,\ddots\,\vdots\,\vdots \\ 0\,0\,0\,\ldots\,1\,0 \\ 0\,0\,0\,\ldots\,0\,0 \end{pmatrix}.$$

Here the first matrix describes the $n$-cycle $(1\,2\,\ldots\,n)$ and the second one the transposition $(1\,2)$. Thus, both generate $S_n$. Thus, we refer to these four elements as $\pi$ (permutation), $\tau$ (transposition), $\nu$ (nondeterminism), and $\epsilon$ (erase). In contrast with the above results, the size of a minimal

[1] Observe, that $B_n^r$ itself is *not* a regular semigroup, since the set of all regular elements in $B_n$ is not closed under multiplication. Observe, that $B_4^r$ contains 63.904 elements, where 40.408 are regular. Note that $B_4$ is of size 65.536. Moreover, the monoid $B_5^r$ is of size 32.311.832, where 8.683.982 elements are regular, and $B_5$ is of size $2^{25} = 33.554.432$.

generating set of $B_n$ grows exponentially with $n$. Devadze [4] stated without a proof that any set consisting of the four generators of $B_n^r$ and representatives of the so called prime $\mathcal{D}$-classes of $B_n$ is a generating set of $B_n$ with the smallest number of elements. Later Devadze's result on the number of generators of $B_n$ was formally proven in [11]; see also, e. g., [8].

**Theorem 2 (Devadze).** *Assume $n \geq 3$. Then the following elements of $B_n$ generate all Boolean matrices of $B_n$ if and only if two of them generate the symmetric group $S_n$, one is similar to the third, another similar to the fourth matrix of generators of the regular semigroup $B_n^r$, and all other elements are any representatives of the prime $\mathcal{D}$-classes of $B_n$. Moreover, no less than these elements generate all of $B_n$.*

A size estimate of $2^{\frac{n^2}{4}-O(n)}$ on the number of generators for $B_n$ can be found in [5, 8, 11]. Also in [5] it was shown that the minimal number of generators of the monoid of $n \times n$ Boolean matrics induces the sequence—this is sequence A346686 of the Online Encyclopedia of Integer Seqeunces (OEIS):

$$2, 3, 5, 7, 13, 68, 2142, 459153, \ldots$$

The size of the largest semigroup generated by *one* $n \times n$ Boolean matrix is studied in [3]. To our knowledge the exact value is unknown, but for $n < 19$ the sizes are $n^2 - 2n + 2$, which gives the OEIS sequence A217990:

$$1, 2, 5, 10, 17, 26, 37, 50, 65, 82, 101, 122, 145, 170, 197, 226, 257, 290, 420, \ldots$$

For $n \geq 19$ the size of the largest semigroup by one $n \times n$ Boolean matrix is only known to be lower bounded by $g(n)$ and upper bounded by $n^2 - 2n + 2 + g(n)$. Also the case of two $n \times n$ Boolean matrix generators were considered in the literature [10], which appears to be quite difficult. Roughly speaking the size of such a semigroup vary greatly with the number of ones in the matrices and can be exponentially. Even the case of three generators was considered, see, e. g., sequence A358784 of OEIS.

Next let us consider the NFA that is induced by the four generators of $B_n^r$ and its equivalent minimal DFA in more detail. To this end define the NFA $A = (Q, \{a, b, c, d\}, \delta, q_0, F)$, where $Q = \{1, 2, \ldots, n\}$, the initial state $q_0 = 1$, the set of final states $F = \{n\}$, and the transition function is given by

- $\delta(i, a) = \{i + 1\}$, for $1 \leq i < n$, and $\delta(n, a) = \{0\}$,
- $\delta(1, b) = \{2\}$, $\delta(2, b) = \{1\}$, and $\delta(i, b) = \{i\}$, for $3 \leq i \leq n$,
- $\delta(1, c) = \{1\}$, $\delta(2, c) = \{1, 2\}$, and $\delta(i, c) = \{i\}$, for $3 \leq i \leq n$, and
- $\delta(i, d) = \{i\}$, for $1 \leq i < n$.

The automaton $A$ is depicted in Figure 2. For this NFA we can prove the following result, for which we introduce the following notation: we refer to the DFA obtained from a finite state device $A = (Q, \Sigma, \delta, q_0, F)$ by the powerset construction[2] as $\mathcal{P}(A) = (2^Q, \Sigma, \delta', \{q_0\}, F')$, where

$$\delta'(P, a) = \cup_{p \in P} \delta(p, a),$$

for $P \in 2^Q$ and $a \in \Sigma$, and $F' = \{ P \in 2^Q \mid P \cap F \neq \emptyset \}$. Here $2^Q$ refers to the powerset of $Q$.

**Theorem 3.** *Let $A = (Q, \{a, b, c, d\}, \delta, q_0, F)$ be the $n$-state NFA defined above. Then the powerset automaton $\mathcal{P}(A)$ is minimal and has $2^n$ initially reachable states.*

[2] Sometimes this construction is also called the subset construction in the literature. The deterministic automaton $\mathcal{P}(A)$ is also called the *powerset automaton of $A$*.

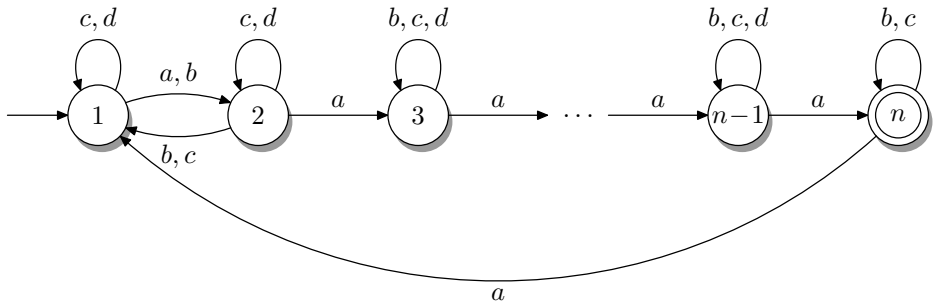


**Fig. 2.** The $n$-state NFA with input letters $a$, $b$, $c$, and $d$ induced by the $n \times n$ Boolean matrics $\pi$, $\tau$, $\nu$, and $\epsilon$, which minimal DFA accepting $L(A)$ requires $2^n$ states.

*Proof.* The upper bound is trivial. For the lower bound we argue as follows: In order to prove the above statement it is sufficient to show that all states of the powerset automaton $B := \mathcal{P}(A)$ are reachable and belong to different equivalence classes with respect to the Myhill-Nerode equivalence relation. Let $\delta'$ refer to the transition function of the powerset automaton.

Let $R$ and $S$ be two distinct states in $2^Q$. Without loss of generality, we assume that state $i$ in $Q$ belongs to $R$ but not to $S$. Since the letter $a$ is a cyclic permutation we find the situation that $n \in \delta'(R, a^{n-i})$ but $n \notin \delta'(S, a^{n-i})$. This shows that $R$ and $S$ are not in the same equivalence class.

Now we are going to show that all states $R$ in $2^Q$ are reachable. Clearly, we do have the situation $\delta'(\{1\}, (ac)^{n-1}) = \{1, 2, \ldots, n\}$. Thus, the full set is reachable. Then we proceed by induction on the number $k := |Q \setminus R|$ of missing elements from $R$ w. r. t. the full set $Q$. Let $k \geq 1$. Consider the state $R = \{i_1, i_2, \ldots, i_{n-k-1}\}$ with $k+1$ missing elements of the powerset automaton. We may assume $1 \leq i_1 < i_2 < \cdots < i_{n-k-1} \leq n$ and let $i$ be any element *not* in $R$. By induction hypothesis the set $S = \{i\} \cup \{i_1, i_2, \ldots i_{n-k-1}\}$, where the union is disjoint, is reachable from the initial state $\{1\}$, since the cardinality of $Q \setminus S$ is $k$. Then $R$ is reachable from $S$ by the input word $z = a^{n-i}da^i$ since

$$\begin{aligned}
\delta'(S, z) &= \delta'(\{i\} \cup \{i_1, i_2, \ldots i_{n-k-1}\}, z) \\
&= \delta'((\{i\} \oplus (n-i)) \cup (\{i_1, i_2, \ldots i_{n-k-1}\} \oplus (n-i)), da^i) \\
&= \delta'(\{n\} \cup (\{i_1, i_2, \ldots i_{n-k-1}\} \oplus (n-i)), da^i) \\
&= \delta'((\{i_1, i_2, \ldots i_{n-k-1}\} \oplus (n-i)), a^i) \\
&= (\{i_1, i_2, \ldots i_{n-k-1}\} \oplus (n-i)) \oplus i \\
&= \{i_1, i_2, \ldots i_{n-k-1}\},
\end{aligned}$$

where $\oplus$ is the operation defined as follows: on a set $R \subseteq Q$ and a number $j \geq 0$ let

$$R \oplus j = \{\, 1 + (i - 1 + j) \bmod n \mid i \in R \,\}.$$

Note that $(R \oplus (n-j)) \oplus j = R$. This proves the stated claim on the reachability of all subsets of $2^Q$. □

Observe, that in the proof of the previous theorem the letter $b$ is not used in the argumentation at all. Moreover, a careful inspection of the previous proof also reveals that the initial state and the singleton set of accepting states can be chosen arbitrarily without changing the statement of Theorem 3. With the next theorem we can conclude that there are $n$-state NFAs, whose NFA-to-DFA conversion is maximal (the minimal DFA requires $2^n$ states), that induce fairly different size syntactic monoids.

The next theorem is a straight-forward observation that relates the algebraic structure induced by the NFA with the syntactic monoid of its accepted language or equivalently with the transition monoid of its equivalent minimal DFA.

**Theorem 4.** *The monoid on $n \times n$ Boolean matrices induced by the NFA $A$ is isomorphic to the syntactic monoid of $L(A)$, if the initially reachable sub-automaton of the DFA $\mathcal{P}(A)$ is minimal.* □

From this theorem we deduce the following corollary:

**Corollary 5.** *For every $n$ there is an $n$-state NFAs that accepts a language of syntactic monoid complexity $2^{n^2}$, which is maximal w. r. t. nondeterministic finite state devices.* □

## 3 A Lower Bound on the NFA-to-DFA Conversion

Theorem 4 can be used to prove a lower bound for the NFA-to-DFA conversion. Let $A$ be an $n$-state NFA that induces a size $r(n)$ monoid of Boolean $n \times n$ matrices. Next assume that the initially reachable sub-automaton of the DFA $\mathcal{P}(A)$ is minimal. Then this sub-automaton has to have a certain size (number of states) in order to induce enough elements in the syntactic monoid or equivalently in the transition monoid of the minimal $m$-state DFA accepting $L(A)$. Hence,

$$m^m \geq r(n) \tag{1}$$

is required. Solving this inequality requires the Lambert $W$-function, which is the solution to $W(x \cdot e^x) = x$. This function is well studied, see, e. g., [1], and satisfies the asymptotics

$$W(x) = \ln x - \ln \ln x + o(1).$$

Next we solve Equation (1) for $m$ as follows:

$$\begin{aligned}
m^m \geq r(n) &\iff m \ln m \geq \ln r(n) && \text{(taking logarithms)}\\
&\iff e^{\ln m} \ln m \geq \ln r(n) && \text{(replace } m \text{ by } e^{\ln m}\text{)}\\
&\iff W(e^{\ln m} \ln m) \geq W(\ln r(n)) && \text{(apply the } W\text{-function)}\\
&\iff \ln m \geq W(\ln r(n)) && \text{(use } W\text{-identity)}\\
&\iff e^{\ln m} \geq e^{W(\ln r(n))} && \text{(take exponentials)}\\
&\iff m \geq e^{W(\ln r(n))} && \text{(replace } e^{\ln m} \text{ by } m\text{).}
\end{aligned}$$

With the asymptotics mentioned above we can estimate

$$m \geq e^{\ln \ln r(n) - \ln \ln \ln r(n) + o(1)}$$

or $m = \Omega(\frac{\ln r(n)}{\ln \ln r(n)})$. Let us summarize:

**Theorem 6.** *Let $A$ be an $n$-state NFA which induces a size $r(n)$ monoid of $n \times n$ Boolean matrices and whose initially reachable sub-automaton of $\mathcal{P}(A)$ is minimal. Then the minimal DFA, which is isomorphic to the initially reachable sub-automaton of $\mathcal{P}(A)$, accepting the language $L(A)$ has $\Omega(\ln r(n)/\ln \ln r(n))$ states.* □

A more precise bound (without $\Omega$-notation) can be given if we use the fact from [7] that

$$\ln x - \ln\ln x + \frac{\ln\ln x}{2\ln x} \leq W(x) \leq \ln x - \ln\ln x + \frac{e}{e-1}\frac{\ln\ln x}{\ln x},$$

for every $x \geq e$, which results in

$$m \geq \frac{\ln r(n)}{\ln\ln r(n)} \cdot e^{\frac{e}{e-1}\frac{\ln\ln\ln r(n)}{\ln\ln r(n)}} = \frac{\ln r(n)}{\ln\ln r(n)} \cdot (\ln\ln r(n))^{\frac{e}{e-1}\frac{1}{\ln\ln r(n)}},$$

for $r(n) \geq e^e \approx 15,1543$. A more readable bound of

$$m \geq \frac{\ln r(n)}{\ln\ln r(n)}(1 + e^{-1}),$$

with the same condition for $r(n)$ follows, if we use, for $x \geq e$, the upper bound estimate

$$W(n) \leq \ln x - \ln\ln x + \ln(1 + e^{-1}),$$

which is also from [7]. It is clear that this lower bound is weak, since the largest $r(n)$ that we can obtain is $r(n) = 2^{n^2}$, which results in a lower bound of

$$\frac{n^2 \ln 2}{2\ln n + \ln\ln 2}.$$

Thus, taking the NFA $A$ induced by the generators of $B_n$, we get this lower bound for NFA-to-DFA conversion, since by Theorem 3 the initially reachable part of the powerset automaton $\mathcal{P}(A)$ is minimal and of maximal size $2^n$. Observe, that $\frac{n^2 \ln 2}{2\ln n + \ln\ln 2} \in o(2^n)$ as $n$ tends to infinity.


## Acknowledgements

Thanks goes to Henning Fernau (Universität Trier, Germany) for some discussion on NFAs and monoids at the beginning of May, 2022. Also thanks to Christian Rauch (Universität Giessen, Germany) for proof reading and to Bianca Truthe (Universität Giessen) for her help preparing this technical report.


## References


1. R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. On the Lambert $W$ function. *Advances in Combinatorial Mathematics*, 5:329–359, December 1996.
2. J. Dénes. On transformations, transformation-semigroups and graphs. In *Theory of Graphs: Proceedings of the Colloquium on Graph Theory*, pages 65–75, Tihany, Hungary, 1968. Academic Press.
3. J. Dénes, K. H. Kim, and F. W. Roush. Automata on one symbol. In *Studies in Pure Mathematics*. Birkhäuser, Basel, 1983.
4. H. M. Devadze. Generating sets of the semigroup of all binary relations on a finite set. *Doklady Akademii Nauk SSSR*, 12:765–768, 1968. In Russian.
5. F. Hivert, J. D. Mitchell, F. L. Smith, and W. A. Wilson. Minimal generating sets for matrix monoids, August 2021.
6. M. Holzer and B. König. On deterministic finite automata and syntactic monoid size. *Theoretical Computer Science*, 327(3):319–347, November 2004.
7. A. Hoorfar and M. Hassani. Inequalities on the Lambert $W$ function and hyperpower function. *Journal of Inequalities in Pure and Applied Mathematics*, 9(2):Article 51, 5pp, 2008.
8. R. Jäschke. Die Struktur der Monoide binärer Relationen auf endlichen Mengen. Diplomarbeit (in German), Technische Universitt Dresden, Fachrichtung Mathematik, Institut für Algebra, April 2005.

9. K. H. Kim and F. W. Roush. On generating regular elements in the semigroup of binary relations. *Semigroup Forum*, 14:29–32, 1977.
10. K. H. Kim and F. W. Roush. Two-generator semigroups of binary relations. *Journal of Mathematical Psychology*, 17:236–246, 1987.
11. J. Konieczny. A proof of Devadzes theorem on generators of the semigroup of boolean matrices. *Semigroup Forum*, 83:281–288, 2011.
12. B. Krawetz, J. Lawrence, and J. Shallit. State complexity and the monoid of transformations of a finite set. *International Journal of Foundations of Computer Science*, 16(3):547–563, Juni 2005.
13. J.-E. Pin. Syntactic semigroups. In G. Rozenberg and A. Salomaa, editors, *Handbook of Formal Languages, Vol. 1, Word, Language, Grammar*, volume 1, pages 679–746. Springer, 1997.
14. J.-É. Pin. Mathematical foundations of automata theory. Unpublished Manuscript, 2022.
15. A. Salomaa. On the composition of functions of several variables ranging over a finite set. *Annales Universitatis Turkuensis*, 41, 1960. Series AI.

A mapping $\alpha : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is referred to as

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ (1)\alpha & (2)\alpha & \ldots & (n)\alpha \end{pmatrix}.$$

In case $(i)\alpha$ is not defined for some $i$ one writes $-$ for $(i)\alpha$ instead. We list minimal generators for $T_n$, for small $n$:

- $n = 1$: permutation $\big(1\big)$.
- $n = 2$: permutations $\big(1\ 2\big)$ and the mapping $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.
- $n \geq 3$: permuations $\big(1\ 2 \ldots n\big)$, $\big(1\ 2\big)$, and the mapping $\begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 1 & 2 & \ldots & n-1 & n-1 \end{pmatrix}$.

From these mappings one can easily construct DFAs, which language has syntactic complexity $2^n$. Recall that the first two permutations generate $S_n$ as mentioned above.
Finally, we list minimal generators for $B_n$, for small $n$:



- $n = 1$: matrices $\big(0\big)$ and $\big(1\big)$.
- $n = 2$: matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
- $n = 3$: matrices $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.
- $n = 4$: matrices $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

For $n = 5$ the number of minimal generators is already 13. From these matrices one can easily construct NFAs, which language has syntactic complexity $2^{n^2}$. Recall that the first four matrices generate $B_n^r$ as mentioned previously.

Institut für Informatik

Justus-Liebig-Universität Giessen
Arndtstr. 2, 35392 Giessen, Germany

## Recent Reports

(Further reports are available at `www.informatik.uni-giessen.de`.)

S. Beier, M. Holzer, *Semi-Linear Lattices and Right One-Way Jumping Finite Automata*, Report 1901, April 2019.

M. Kutrib, T. Worsch, *Self-Verifying Cellular Automata*, Report 1803, April 2018.

S. Beier, M. Holzer, *Properties of Right One-Way Jumping Finite Automata*, Report 1802, March 2018.

B. Truthe, *Hierarchy of Subregular Language Families*, Report 1801, February 2018.

M. Holzer, M. Hospodár, *On the Magic Number Problem of the Cut Operation*, Report 1703, October 2017.

M. Holzer, S. Jakobi, *A Note on the Computational Complexity of Some Problems for Self-Verifying Finite Automata*, Report 1702, April 2017.

S. Beier, M. Holzer, M. Kutrib, *On the Descriptional Complexity of Operations on Semilinear Sets*, Report 1701, April 2017.

M. Holzer, S. Jakobi, M. Wendlandt, *On the Computational Complexity of Partial Word Automata Problems*, Report 1404, May 2014.

H. Gruber, M. Holzer, *Regular Expressions From Deterministic Finite Automata, Revisited*, Report 1403, May 2014.

M. Kutrib, A. Malcher, M. Wendlandt, *Deterministic Set Automata*, Report 1402, April 2014.

M. Holzer, S. Jakobi, *Minimal and Hyper-Minimal Biautomata*, Report 1401, March 2014.

J. Kari, M. Kutrib, A. Malcher (Eds.), *19th International Workshop on Cellular Automata and Discrete Complex Systems AUTOMATA 2013 Exploratory Papers*, Report 1302, September 2013.

M. Holzer, S. Jakobi, *Minimization, Characterizations, and Nondeterminism for Biautomata*, Report 1301, April 2013.

A. Malcher, K. Meckel, C. Mereghetti, B. Palano, *Descriptional Complexity of Pushdown Store Languages*, Report 1203, May 2012.

M. Holzer, S. Jakobi, *On the Complexity of Rolling Block and Alice Mazes*, Report 1202, March 2012.

M. Holzer, S. Jakobi, *Grid Graphs with Diagonal Edges and the Complexity of Xmas Mazes*, Report 1201, January 2012.

H. Gruber, S. Gulan, *Simplifying Regular Expressions: A Quantitative Perspective*, Report 0904, August 2009.

M. Kutrib, A. Malcher, *Cellular Automata with Sparse Communication*, Report 0903, May 2009.

M. Holzer, A. Maletti, *An $n \log n$ Algorithm for Hyper-Minimizing States in a (Minimized) Deterministic Automaton*, Report 0902, April 2009.

H. Gruber, M. Holzer, *Tight Bounds on the Descriptional Complexity of Regular Expressions*, Report 0901, February 2009.

M. Holzer, M. Kutrib, and A. Malcher (Eds.), *18. Theorietag Automaten und Formale Sprachen*, Report 0801, September 2008.

M. Holzer, M. Kutrib, *Flip-Pushdown Automata: Nondeterminism is Better than Determinism*, Report 0301, February 2003

M. Holzer, M. Kutrib, *Flip-Pushdown Automata: $k+1$ Pushdown Reversals are Better Than $k$*, Report 0206, November 2002

M. Holzer, M. Kutrib, *Nondeterministic Descriptional Complexity of Regular Languages*, Report 0205, September 2002

H. Bordihn, M. Holzer, M. Kutrib, *Economy of Description for Basic Constructions on Rational Transductions*, Report 0204, July 2002

M. Kutrib, J.-T. Löwe, *String Transformation for n-dimensional Image Compression*, Report 0203, May 2002

A. Klein, M. Kutrib, *Grammars with Scattered Nonterminals*, Report 0202, February 2002

A. Klein, M. Kutrib, *Self-Assembling Finite Automata*, Report 0201, January 2002

M. Holzer, M. Kutrib, *Unary Language Operations and its Nondeterministic State Complexity*, Report 0107, November 2001

A. Klein, M. Kutrib, *Fast One-Way Cellular Automata*, Report 0106, September 2001

M. Holzer, M. Kutrib, *Improving Raster Image Run-Length Encoding Using Data Order*, Report 0105, July 2001