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SEMI-LINEAR LATTICES AND RIGHT ONE-WAY JUMPING FINITE AUTOMATA

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Semi-Linear Lattices and Right One-Way Jumping Finite Automata

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Abstract. Right one-way jumping automata (ROWJFAs) are an automaton model that was recently introduced for processing the input in a discontinuous way. In [S. Beier, M. Holzer: Properties of right one-way jumping finite automata. In Proc. 20th DCFS, number 10952 in LNCS, 2018] it was shown that the permutation closed languages accepted by ROWJFAs are exactly that with a finite number of positive Myhill-Nerode classes. Here a Myhill-Nerode equivalence class $[w]_L$ of a language L is said to be positive if w belongs to L. Obviously, this notion of positive Myhill-Nerode classes generalizes to sets of vectors of natural numbers. We give a characterization of the linear sets of vectors with a finite number of positive Myhill-Nerode classes, which uses rational cones. Furthermore, we investigate when a set of vectors can be decomposed as a finite union of sets of vectors with a finite number of positive Myhill-Nerode classes. A crucial role play lattices, which are special semi-linear sets that are defined as a natural way to extend "the pattern" of a linear set to the whole set of vectors of natural numbers in a given dimension. We show deep connections of lattices to the Myhill-Nerode relation and to rational cones. Some of these results will be used to give characterization results about ROWJFAs with multiple initial states. For binary alphabets we show connections of these and related automata to counter automata.

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68Q05 Models of computation
68Q10 Modes of computation
68Q45 Formal languages and automata

Additional Key Words and Phrases: Right One-Way Jumping Finite Automata, Semi-Linear Sets and Lattices, Rational Cones, Myhill-Nerode Classes, Permutation Closed Languages.

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1 Introduction

Semi-linear sets, Presburger arithmetic, and context-free languages are closely related to each other by the famous results of Ginsburg and Spanier [11] and Parikh [15]. More precisely, a set is semi-linear if and only it is expressible in Presburger arithmetic, which is the first order theory of addition. These sets coincide with the Parikh images of regular languages, which are exactly the same as the Parikh images of context-free languages by Parikh's theorem that states that the Parikh image of any context-free language is semi-linear. Since then semi-linear sets and results thereof are well known in computer science. Recently, the interest on semi-linear sets has increased significantly. On the one hand, there was renewed interest in equivalence problems on commutative languages [13] which obviously correspond to their Parikh-image, and on the other hand, it turned out that semi-linearity is the key to understand the accepting power of jumping finite automata, an automaton model that was introduced in [14] for discontinuous information processing. Roughly speaking, a jumping finite automaton is an ordinary finite automaton, which is allowed to read letters from anywhere in the input string, not necessarily only from the left of the remaining input. Moreover, semi-linear sets were also subject to descriptional complexity considerations in [3] and [5].

The tight relation between semi-linear sets and jumping automata is not limited to this automaton model, but also turns over to right one-way jumping automata as shown in [1, 2]. Right one-way jumping automata (ROWJFAs) were introduced in [4] and allows the device to process the input also in a discontinuous way with the restriction that the input head reads deterministically from left-to-right starting from the leftmost letter in the input and when it reaches the end of the input word, it returns to the beginning and continues the computation. Most questions on formal language related problems such as inclusion problems, closure properties, and decidability of standard problems concerning ROWJFAs were answered recently in one of the papers [1, 2, 4]. One of the main results on these devices was a characterization of the induced language family that reads as follows: a permutation closed language L belongs to **ROWJ**, the family of all languages accepted by ROWJFAs, if and only if L can be written as the *finite union* of Myhill-Nerode equivalence classes. Observe, that the overall number of equivalence classes can be infinite. This result nicely contrasts the characterization of regular languages, which requires that the overall number of equivalence classes is finite.

In this paper we deepen the understanding of the Myhill-Nerode equivalence relation given by a subset of $\mathbb{N}^k$ as defined in [10]. For a subset $S \subseteq \mathbb{N}$ and the induced Myhill-Nerode relation, an equivalence class is called positive if and only if the vectors of the class lie in S. We characterize in which cases linear sets have only a finite number of positive equivalence classes in terms of rational cones, which are a special type of convex cones that are important objects in different areas of mathematics and computer science like combinatorial commutative algebra, geometric combinatorics, and integer programming. A special type of semi-linear sets called lattices is introduced. Their definition is inspired by the mathematical object of a lattice which is of great importance in geometry and group theory, see [6]. These lattices are subgroups of $\mathbb{R}^k$ that are isomorphic to $\mathbb{Z}^k$ and span the real vector space $\mathbb{R}^k$. Our semi-linear lattices are defined like linear sets, but allowing integer coefficients for the period vectors, instead of only natural numbers. However our lattices are still, per definition, subsets of $\mathbb{N}^k$. Lattices have only one positive Myhill-Nerode class and can be decomposed as a finite union of linear sets with only one positive Myhill-Nerode class. We give a characterization of the lattices that can even be decomposed as a finite union of linear sets with linearly independent period sets and only one positive Myhill-Nerode class and again get a connection to rational cones. That is why we study these objects in more detail and show that the set of vectors with only non-negative components in a linear subspace of dimension n of $\mathbb{R}^k$ spanned by a subset of $\mathbb{N}^k$ always forms a rational cone spanned by a linearly independent subset of $\mathbb{N}^k$ if and only if $n \in \{0, 1, 2, k\}$. These result has consequences for the mentioned decompositions of lattices. We show when a subset of $\mathbb{N}^k$ can be decomposed as a finite union of those subsets that have only a finite number of positive Myhill-Nerode classes. That result heavily depends on the theory of lattices.

The obtained results on lattices are applied to ROWJFAs generalized to devices with multiple initial states (MROWJFAs). This slight generalization is in the same spirit as the one for ordinary finite automata that leads to multiple entry deterministic finite automata [8]. We show basic properties of MROWJFAs and inclusion relations to families of the Chomsky hierarchy and related families. A deep connection between the family of permutation closed languages accepted by MROWJFAs (the corresponding language family is referred to **pMROWJ**) and lattices is shown. This connection allows us to deduce a characterization of languages in **pMROWJ** from our results about lattices and decompositions of subsets of $\mathbb{N}^k$. We also investigate the languages accepted by MROWJFAs and related languages families for the special case of a binary input alphabet and get in some cases different or stronger results than for arbitrary alphabets. We can show that each permutation closed semi-linear language (these are exactly the languages accepted by jumping finite automata) over a binary alphabet is a accepted by a counter automaton. Furthermore, each language over a binary alphabet accepted by a ROWJFA is also accepted by a realtime deterministic counter automaton. Our results for lattices lead to a characterization, which is stronger than the one for arbitrary alphabets, of the languages over binary alphabets in **pMROWJ**: these are exactly the languages which are a finite union of permutation closed languages accepted by a ROWJFA, which are characterized by positive Myhill-Nerode classes as stated above.

2 Preliminaries

We use $\subseteq$ for inclusion and $\subset$ for proper inclusion of sets. For a binary relation $\sim$ let $\sim^+$ and $\sim^*$ denote the transitive closure of $\sim$ and the transitive-reflexive closure of $\sim$, respectively. In the standard manner, $\sim$ is extended to $\sim^n$, where $n \geq 0$. Let $\mathbb{Z}$ be the set of integers, $\mathbb{Q}$ be the set of rational numbers, $\mathbb{R}$ be the set of real numbers, and $\mathbb{N}$ ($\mathbb{Q}_{\geq 0}$, $\mathbb{R}_{\geq 0}$, respectively) be the set of integers (rational numbers, real numbers, respectively) which are non-negative. Let $k \geq 0$. For a set $T \subseteq \{1, 2, \ldots, k\}$ with $T = \{t_1, t_2, \ldots, t_\ell\}$ and $t_1 < t_2 < \cdots < t_\ell$ we define $\pi_{k,T} : \mathbb{N}^k \to \mathbb{N}^{|T|}$ as

$$\pi_{k,T}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k) = (\boldsymbol{x}_{t_1}, \boldsymbol{x}_{t_2}, \ldots, \boldsymbol{x}_{t_{|T|}}).$$

The elements of $\mathbb{R}^k$ can be partially ordered by the $\leq$-relation on vectors. For vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k$ we write $\boldsymbol{x} \leq \boldsymbol{y}$ if all components of $\boldsymbol{x}$ are less or equal to the corresponding components of $\boldsymbol{y}$. The value $||\boldsymbol{x}||_1$ is the *taxicab norm* of $\boldsymbol{x}$, that is,

$$||(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k)||_1 = \sum_{i=1}^{k} |\boldsymbol{x}_i|,$$

and the value $||\boldsymbol{x}||_2$ is the *Euclidean norm* of $\boldsymbol{x}$, that is,

$$||(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k)||_2 = \sqrt{\sum_{i=1}^{k} \boldsymbol{x}_i^2}.$$

For a set $S \subseteq \mathbb{R}^k$ let $\mathsf{span}(S)$ be the intersection of all linear subspaces of $\mathbb{R}^k$ that are supersets of S, which vector space is also called the *linear subspace of* $\mathbb{R}^k$ *spanned by* S. For a linear subspace V of $\mathbb{R}^k$ let $\dim(V)$ be the dimension of V. For a finite $S \subseteq \mathbb{Z}^k$ the *rational cone spanned by* S is $\mathsf{cone}(S) = \{ \sum_{\boldsymbol{x_i} \in S} \lambda_i \cdot \boldsymbol{x_i} \mid \lambda_i \in \mathbb{R}_{\geq 0} \} \subseteq \mathbb{R}^k$. A *linearly independent rational cone* in $\mathbb{R}^k$ is a set of the form $\mathsf{cone}(S)$ for a linearly independent $S \subseteq \mathbb{Z}^k$. Each rational cone is a finite union of linearly independent rational cones, see for example [16]. For $r \in \mathbb{R}_{\geq 0}$ define $B_r(\boldsymbol{x}) = \{ \boldsymbol{z} \in \mathbb{R}^k \mid ||\boldsymbol{z} - \boldsymbol{x}||_2 \leq r \}$, the *ball of radius* r *around* $\boldsymbol{x}$, and for a set $S \subseteq \mathbb{R}^k$ and $\lambda \in \mathbb{R}$ let $\lambda S = \{ \lambda \boldsymbol{z} \mid \boldsymbol{z} \in S \}$ and $S + \boldsymbol{x} = \{ \boldsymbol{z} + \boldsymbol{x} \mid \boldsymbol{z} \in S \}$.

For a $\boldsymbol{c} \in \mathbb{N}^k$ and a finite $P \subseteq \mathbb{N}^k$ let

$$\mathsf{L}(\boldsymbol{c}, P) = \left\{ \boldsymbol{c} + \sum_{\boldsymbol{x_i} \in P} \lambda_i \cdot \boldsymbol{x_i} \;\middle|\; \lambda_i \in \mathbb{N} \right\} \quad \text{and} \quad \mathsf{La}(\boldsymbol{c}, P) = \left\{ \boldsymbol{c} + \sum_{\boldsymbol{x_i} \in P} \lambda_i \cdot \boldsymbol{x_i} \;\middle|\; \lambda_i \in \mathbb{Z} \right\} \cap \mathbb{N}^k.$$

By definition, $\mathsf{L}(\boldsymbol{c}, P) \subseteq \mathsf{La}(\boldsymbol{c}, P)$. The vector c is called the *constant vector* whereas the set P is called the set of *periods* of $\mathsf{L}(\boldsymbol{c}, P)$ and of $\mathsf{La}(\boldsymbol{c}, P)$. Sets of the form $\mathsf{L}(\boldsymbol{c}, P)$, for a $\boldsymbol{c} \in \mathbb{N}^k$ and a finite $P \subseteq \mathbb{N}^k$, are called *linear* subsets of $\mathbb{N}^k$, while sets of the form $\mathsf{La}(\boldsymbol{c}, P)$ are called *lattices.* A subset of $\mathbb{N}^k$ is said to be *semi-linear* if it is a finite union of linear subsets. For a $\boldsymbol{c} \in \mathbb{N}^k$, a natural number $n \geq 0$, and $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n} \in \mathbb{N}^k$ we have that $\mathsf{La}\left(\boldsymbol{c}, \{\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n}\}\right)$ is equal to the set of all $\boldsymbol{y} \in \mathbb{N}^k$ such that there exists $\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_n, \mu_n \in \mathbb{N}$ with $\boldsymbol{c} + \sum_{i=1}^n \lambda_i \boldsymbol{x_i} = \boldsymbol{y} + \sum_{i=1}^n \mu_i \boldsymbol{x_i}$, which is a Presburger set. Since the Presburger sets are exactly the semi-linear sets by [11], every lattice is semi-linear. In order to explain our definitions we give an example.

Example 1. Consider the vector $\boldsymbol{c} = (4, 4)$ and the period vectors $\boldsymbol{p_1} = (1, 2)$ and $\boldsymbol{p_2} = (2, 0)$. A graphical presentation of the linear set $\mathsf{L}(\boldsymbol{c}, P)$ with $P = \{\boldsymbol{p_1}, \boldsymbol{p_2}\}$ is given on the left of Figure 1. The constant vector $\boldsymbol{c}$ is drawn red and both periods $\boldsymbol{p_1}$ and $\boldsymbol{p_2}$ are depicted in blue. The black dots indicate the elements that belong to $\mathsf{L}(\boldsymbol{c}, P)$. The lattice $\mathsf{La}(\boldsymbol{c}, P)$ is drawn in

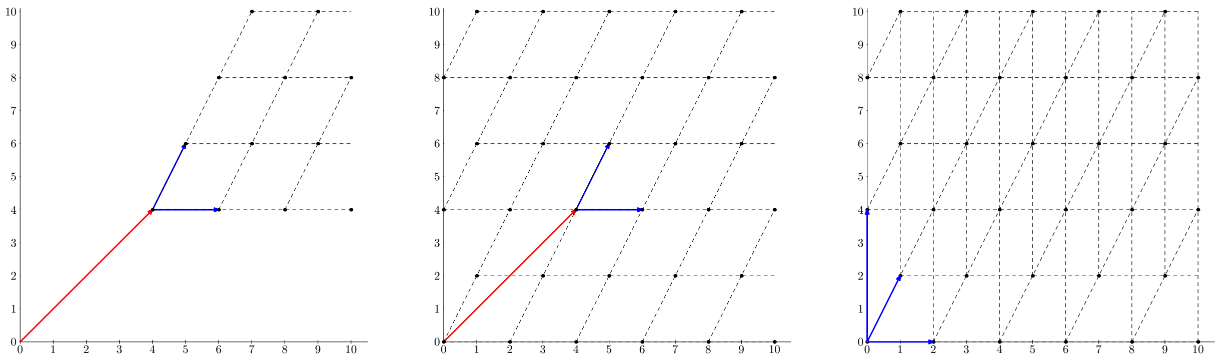

Fig. 1. The linear set $\mathsf{L}(\boldsymbol{c}, P)$ with $\boldsymbol{c} = (4, 4)$ and $P = \{\boldsymbol{p_1}, \boldsymbol{p_2}\}$, where $\boldsymbol{p_1} = (1, 2)$ and $\boldsymbol{p_2} = (2, 0)$ drawn on the left. The black dots indicate membership in $\mathsf{L}(\boldsymbol{c}, P)$. The lattice $\mathsf{La}(\boldsymbol{c}, P)$ is depicted in the middle. Here the black dots refer to membership in $\mathsf{La}(\boldsymbol{c}, P)$. On the right a representation of $\mathsf{La}(\boldsymbol{c}, P)$ as a linear set is shown. Again, the constant vector $\mathbf{0}$ is shown as a red dot and the period vector are colored blue.

the middle of Figure 1. Again, the constant vector is colored red, while both periods are in blue. Since now integer coefficients are allowed, there are new elements compared to $\mathsf{L}(\boldsymbol{c}, P)$ that belong to $\mathsf{La}(\boldsymbol{c}, P)$. On the right of Figure 1 it is shown that the set $\mathsf{La}(\boldsymbol{c}, P)$ can be written as a linear set by using the constant vector $\mathbf{0}$ and the three period vectors drawn in blue, that is, $\mathsf{La}(\boldsymbol{c}, P) = \mathsf{L}(\mathbf{0}, \{\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}\})$, where $\boldsymbol{p_3} = (0, 2)$.

An important result about semi-linear sets is that each semi-linear set can be written as a finite union of linear sets with linearly independent period sets [9]:

Theorem 2. *Let $k \geq 0$ and $S \subseteq \mathbb{N}^k$ be a semi-linear set. Then, there is $m \geq 0$, vectors $\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_m} \in \mathbb{N}^k$, and linearly independent $P_1, P_2, \ldots, P_m \subseteq \mathbb{N}^k$ such that $S = \bigcup_{i=1}^{m} \mathsf{L}(\boldsymbol{c_i}, P_i)$.*

Now, we recall some basic definitions from formal language theory. Let Σ be an alphabet. Then Σ^* is the set of all words over Σ, including the empty word λ. For a language $L \subseteq \Sigma^*$ define the set $\mathsf{perm}(L) = \cup_{w \in L} \mathsf{perm}(w)$, where $\mathsf{perm}(w) = \{ v \in \Sigma^* \mid v \text{ is a permutation of } w \}$. A language L is called *permutation closed* if $L = \mathsf{perm}(L)$. The length of a word $w \in \Sigma^*$ is denoted by $|w|$. For the number of occurrences of a symbol a in w we use the notation $|w|_a$. If Σ is the ordered alphabet $\Sigma = \{a_1, a_2, \ldots, a_k\}$, the *Parikh-mapping* $\psi : \Sigma^* \to \mathbb{N}^k$ is the function $w \mapsto (|w|_{a_1}, |w|_{a_2}, \ldots, |w|_{a_k})$. The set $\psi(L)$ is called the *Parikh-image* of L. A language $L \subseteq \Sigma^*$ is called *semi-linear* if its Parikh-image $\psi(L)$ is a semi-linear set.

For a language $L \subseteq \Sigma^*$ let $\sim_L$ be the *Myhill-Nerode equivalence relation* on Σ^*. So, for $v, w \in \Sigma^*$, we have $v \sim_L w$ if and only if, for all $u \in \Sigma^*$, the equivalence $vu \in L \Leftrightarrow wu \in L$ holds. For $w \in \Sigma^*$, we call the equivalence class $[w]_{\sim_L}$ *positive* if and only if $w \in L$. For $k \geq 0$ and a subset $S \subseteq \mathbb{N}^k$ the equivalence relation $\equiv_S$ on $\mathbb{N}^k$ was defined in [10] as follows: for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}^k$, we have $\boldsymbol{x} \equiv_S \boldsymbol{y}$ if and only if $(\boldsymbol{x} + \boldsymbol{z}) \in S \Leftrightarrow (\boldsymbol{y} + \boldsymbol{z}) \in S$, for all $\boldsymbol{z} \in \mathbb{N}^k$. For $\boldsymbol{x} \in \mathbb{N}^k$, we call the equivalence class $[\boldsymbol{x}]_{\equiv_S}$ *positive* if and only if $\boldsymbol{x} \in S$. We will also refer to $\equiv_S$ as the Myhill-Nerode equivalence relation. If $L \subseteq \Sigma^*$ is a permutation closed language and $v, w \in L$ we have $v \sim_L w$ if and only if $\psi(v) \equiv_{\psi(L)} \psi(w)$. So, the language L is regular if and only if $\mathbb{N}^{|\Sigma|}/ \equiv_{\psi(L)}$ is finite.

Let **REG**, **DCF**, **CF**, and **CS** be the families of regular, deterministic context-free, context-free, and context-sensitive languages. Moreover, we are interested in families of permutation closed languages. These language families are referred to by a prefix **p**. E.,g., **pREG** denotes the language family of all permutation closed regular languages. Let **JFA** be the family of all languages accepted by jumping finite automata, see [14]. These are exactly the permutation closed semi-linear languages.

A *right one-way jumping finite automaton with multiple initial states* (MROWJFA) is a tuple $A = (Q, \Sigma, R, S, F)$, where Q is the *finite set of states*, Σ is the *finite input alphabet*, R is a *partial function* from $Q \times \Sigma$ to Q, $S \subseteq Q$ is the *set of initial or start states*, and $F \subseteq Q$ is t he *set of final states*. A *configuration* of A is a string in $Q\Sigma^*$. The *right one-way jumping relation*, symbolically denoted by $\circlearrowright_A$ or just $\circlearrowright$ if it is clear which MROWJFA we are referring to, over $Q\Sigma^*$ is defined as follows. For $p \in Q$ we set

$$\Sigma_{R,p} = \{ b \in \Sigma \mid R(p, b) \text{ is defined} \},$$

and if there is no danger of confusion we simple refer to $\Sigma_{R,p}$ as Σ_p. Let $p, q \in Q$, $a \in \Sigma$, $w \in \Sigma^*$. If $R(p, a) = q$, then we have $paw \circlearrowright qw$; otherwise, in case $R(p, a)$ is undefined, we get $paw \circlearrowright pwa$. So, the automaton jumps over a symbol, when it cannot be read. The *language accepted* by A is

$$L_R(A) = \{ w \in \Sigma^* \mid \exists s \in S, f \in F : sw \circlearrowright^* f \}.$$

We say that A *accepts* $w \in \Sigma^*$ if $w \in L_R(A)$ and that A *rejects* w otherwise. Let **MROWJ** be the family of all languages that are accepted by MROWJFAs. Furthermore, in case the MROWJFA has a single initial state, i.e., $|S| = 1$, then we simply speak of a right one-way jumping automaton (ROWJFA) and refer to the family of languages accepted by ROWJFAs by **ROWJ**. Obviously, by definition we have **ROWJ** $\subseteq$ **MROWJ**. We given an example of a ROWJFA:

Example 3. Let A be the ROWJFA $A = (\{q_0, q_1, q_2, q_3\}, \{a, b\}, R, q_0, \{q_3\})$, where the set R consists of the rules $q_0b \to q_1$, $q_0a \to q_2$, $q_2b \to q_3$, and $q_3a \to q_2$. The automaton A is

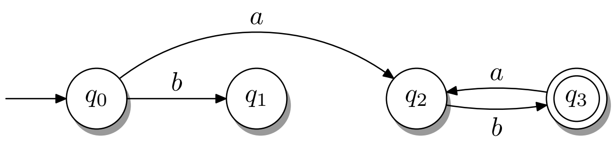

Fig. 2. The ROWJFA A.

depicted in Figure 2. To show how ROWJFAs work, we give an example computation of A on the input $aabbba$:

$$q_0aabbba \circlearrowright q_2abbba \circlearrowright^2 q_3bbaa \circlearrowright^3 q_2abb \circlearrowright^2 q_3ba \circlearrowright^2 q_2b \circlearrowright q_3$$

That shows $aabbba \in L_R(A)$. Analogously, one can see that every word that contains the same number of a's and b's and that begins with an a is in $L_R(A)$. On the other hand, no other word can be accepted by A, interpreted as an ROWJFA. So, we get $L_R(A) = \{ w \in a\{a,b\}^* \mid |w|_a = |w|_b \}$. Notice that this language is non-regular and not closed under permutation.

The following characterization of permutation closed languages accepted by ROWJFAs is known from [2].

Theorem 4. *Let L be a permutation closed language. Then, the language L is in* **pROWJ** *if and only if the Myhill-Nerode relation $\sim_L$ has only a finite number of positive equivalence classes.*

3 Lattices, Linear Sets, and Myhill-Nerode Classes

Because of Theorem 4 a permutation closed language is in **ROWJ** if and only if the Parikh-image has only a finite number of positive Myhill-Nerode equivalence classes. In this section we will study these kind of subsets of $\mathbb{N}^k$. Linear sets and lattices will play a key role in our theory. We will investigate decompositions of subsets of $\mathbb{N}^k$ as finite unions of such subsets that have only a finite number of positive equivalence classes. This will lead to characterization results about the language class **pMROWJ** in the next section.

3.1 Connections Between Linear Sets and Rational Cones

It was pointed out in [12] that "rational cones in $\mathbb{R}^d$ are important objects in toric algebraic geometry, combinatorial commutative algebra, geometric combinatorics, integer programming." In the following we will see how rational cones are related to the property of linear sets to have only a finite number of positive Myhill-Nerode equivalence classes. The next property of linear sets is straightforward.

Lemma 5. *For $k \geq 0$, vectors $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{N}^k$, and a finite set $P \subseteq \mathbb{N}^k$ the map $\mathsf{L}(\boldsymbol{c}, P) \to \mathsf{L}(\boldsymbol{d}, P)$ given by $\boldsymbol{x} \mapsto \boldsymbol{x} - \boldsymbol{c} + \boldsymbol{d}$ induces a bijection from $\mathsf{L}(\boldsymbol{c}, P)/\equiv_{\mathsf{L}(\boldsymbol{c},P)}$ to $\mathsf{L}(\boldsymbol{d}, P)/\equiv_{\mathsf{L}(\boldsymbol{d},P)}$.*

Now, we define two properties of subsets of $\mathbb{N}^k$ which involve rational cones. Let $k \geq 0$ and $S \subseteq \mathbb{N}^k$. Then, the set S has the *linearly independent rational cone property* if and only if $\mathsf{span}(S) \cap (\mathbb{R}_{\geq 0})^k = \mathsf{cone}(T)$, for some a linearly independent $T \subseteq \mathbb{N}^k$. The set S has the *own rational cone property* if and only if S is finite and it holds $\mathsf{span}(S) \cap (\mathbb{R}_{\geq 0})^k = \mathsf{cone}(S)$.

A linear set has only a finite number of positive Myhill-Nerode equivalence classes if and only if the period set has the own rational cone property:

Theorem 6. *Let $k \geq 0$ and $P \subseteq \mathbb{N}^k$ be finite. Then, $\left|\mathsf{L}(\mathbf{0}, P)/\equiv_{\mathsf{L}(\mathbf{0},P)}\right| < \infty$ if and only if P has the own rational cone property.*

Proof. Let $P = \{\boldsymbol{p_1}, \boldsymbol{p_2}, \ldots, \boldsymbol{p_{|P|}}\}$. Assume that P has the own rational cone property. Let U be the finite set

$$U = \left\{ \sum_{i=1}^{|P|} \nu_i \boldsymbol{p_i} \;\middle|\; \nu_i \in \mathbb{R},\, 0 \leq \nu_i < 1 \right\} \cap \mathbb{N}^k.$$

Let $\boldsymbol{u} \in U$ and

$$V_{\boldsymbol{u}} = \left\{ (\xi_1, \xi_2, \ldots, \xi_{|P|}) \in \mathbb{N}^{|P|} \;\middle|\; \boldsymbol{u} + \sum_{i=1}^{|P|} \xi_i \boldsymbol{p_i} \in \mathsf{L}(\mathbf{0}, P) \right\}.$$

Consider the $\leq$-relation on $\mathbb{N}^{|P|}$. For all $\boldsymbol{v} \in V_{\boldsymbol{u}}$ and $\boldsymbol{w} \in \mathbb{N}^{|P|}$ with $\boldsymbol{v} \leq \boldsymbol{w}$ we have $\boldsymbol{w} \in V_{\boldsymbol{u}}$. Let $W_{\boldsymbol{u}}$ be the set of minimal elements of $V_{\boldsymbol{u}}$. Due to [7] each subset of $\mathbb{N}^{|P|}$ has only a finite number of minimal elements, so $W_{\boldsymbol{u}}$ is finite. For $\boldsymbol{x} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_{|P|}}) \in \mathbb{N}^{|P|}$ let $X_{\boldsymbol{u},\boldsymbol{x}}$ be the finite set that contains the elements

$$(\max(0, \xi_1 - \boldsymbol{x_1}), \max(0, \xi_2 - \boldsymbol{x_2}), \ldots, \max(0, \xi_{|P|} - \boldsymbol{x_{|P|}})),$$

where $(\xi_1, \xi_2, \ldots, \xi_{|P|}) \in W_{\boldsymbol{u}}$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}^{|P|}$ we have $\boldsymbol{x} + \boldsymbol{y} \in V_{\boldsymbol{u}}$ if and only if there is a $\boldsymbol{z} \in X_{\boldsymbol{u},\boldsymbol{x}}$ with $\boldsymbol{z} \leq \boldsymbol{y}$. The set $\{\, X_{\boldsymbol{u},\boldsymbol{x}} \mid \boldsymbol{x} \in \mathbb{N}^{|P|} \,\}$ is finite.

Let now $t \in \mathsf{L}(\mathbf{0}, P)$ and $t' \in \mathbb{N}^k$. If $\boldsymbol{t'} \notin \mathsf{span}(P)$, we have $\boldsymbol{t}+\boldsymbol{t'} \notin \mathsf{L}(\mathbf{0}, P)$. So, let $\boldsymbol{t'} \in \mathsf{span}(P)$. There is an $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{|P|}) \in \mathbb{N}^{|P|}$ with $\boldsymbol{t} = \sum_{i=1}^{|P|} \boldsymbol{x}_i \boldsymbol{p_i}$. Because P has the own rational cone property, there are

$$\boldsymbol{u'} \in U \quad \text{and} \quad \boldsymbol{x'} = (\boldsymbol{x'}_1, \boldsymbol{x'}_2, \ldots, \boldsymbol{x'}_{|P|}) \in \mathbb{N}^{|P|}$$

such that $\boldsymbol{t'} = \boldsymbol{u'} + \sum_{i=1}^{|P|} \boldsymbol{x'}_i \boldsymbol{p_i}$. We have $\boldsymbol{t} + \boldsymbol{t'} \in \mathsf{L}(\mathbf{0}, P)$ if and only if $\boldsymbol{x} + \boldsymbol{x'} \in V_{\boldsymbol{u'}}$, which holds if and only if there is a $\boldsymbol{z} \in X_{\boldsymbol{u'},\boldsymbol{x}}$ with $\boldsymbol{z} \leq \boldsymbol{x'}$. So, the set

$$\left\{ \boldsymbol{r} \in \mathbb{N}^k \;\middle|\; \boldsymbol{t} + \boldsymbol{r} \in \mathsf{L}(\mathbf{0}, P) \right\}$$

only depends on the map

$$U \to \left\{ X_{\boldsymbol{v},\boldsymbol{y}} \;\middle|\; \boldsymbol{v} \in U, \boldsymbol{y} \in \mathbb{N}^{|P|} \right\}$$

given by $\boldsymbol{v} \mapsto X_{\boldsymbol{v},\boldsymbol{x}}$. That shows $\left|\mathsf{L}(\mathbf{0}, P)/\equiv_{\mathsf{L}(\mathbf{0},P)}\right| < \infty$.

Assume now that P does not have the own rational cone property. This gives us the inclusion $\mathsf{cone}(P) \subset \mathsf{span}(P) \cap (\mathbb{R}_{\geq 0})^k$. So, let $\boldsymbol{x} \in \mathsf{span}(P) \cap (\mathbb{R}_{\geq 0})^k \setminus \mathsf{cone}(P)$ and

$$T = \left\{ i \in \{1, 2, \ldots, k\} \;\middle|\; \pi_{k,\{i\}}(\boldsymbol{x}) = 0 \right\}.$$

By basic linear algebra there is a non-empty linearly independent $Q \subset \mathbb{Q}^k$ with

$$\mathsf{span}(Q) = \{\, \boldsymbol{y} \in \mathsf{span}(P) \mid \pi_{k,T}(\boldsymbol{y}) = \mathbf{0} \,\}.$$

Because $\boldsymbol{x} \in \mathsf{span}(Q) \cap (\mathbb{R}_{\geq 0})^k \setminus \mathsf{cone}(P)$ and $\mathsf{cone}(P)$ is a topological closed set, see [16], there is a

$$\boldsymbol{z} \in \mathsf{span}(Q) \cap (\mathbb{Q}_{\geq 0})^k \setminus \mathsf{cone}(P) \subseteq \mathsf{span}(P) \cap (\mathbb{Q}_{\geq 0})^k \setminus \mathsf{cone}(P).$$

So, there are $\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_{|P|}, \mu_{|P|} \in \mathbb{Q}_{\geq 0}$ with

$$\sum_{i=1}^{|P|} (\lambda_i - \mu_i)\boldsymbol{p_i} \in (\mathbb{Q}_{\geq 0})^k \setminus \mathsf{cone}(P).$$

We may assume that the values $\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_{|P|}, \mu_{|P|} \in \mathbb{N}$ by multiplying with the lowest common denominator. Let

$$R = \min \left\{ r \in \mathbb{R} \;\middle|\; \sum_{i=1}^{|P|} (\lambda_i + r\mu_i)\boldsymbol{p_i} \in \mathsf{cone}(P) \right\} \in (-1, 0].$$

We refer to M as the set of minimal elements regarding the $\leq$-relation on $\mathbb{N}^k$ of the set $\mathbb{N}^k \cap \mathsf{span}(P) \setminus \{\boldsymbol{0}\}$. Due to [7] the set M is finite. We have

$$\mathbb{N}^k \cap \mathsf{span}(P) = \mathsf{L}(\boldsymbol{0}, M).$$

For each linearly independent $B \subseteq P$ with $\mathsf{span}(B) = \mathsf{span}(P)$ let N_B be the lowest common multiply of the numbers $\min\{\, n \in \mathbb{N} \setminus \{0\} \mid n\boldsymbol{v} \in \mathsf{La}(\boldsymbol{0}, B) \,\}$, for $\boldsymbol{v} \in M$. This gives us

$$\mathsf{L}(\boldsymbol{0}, B) \cap (N_B \cdot \mathbb{N}^k) = \mathsf{cone}(B) \cap (N_B \cdot \mathbb{N}^k),$$

for each linearly independent $B \subseteq P$ with $\mathsf{span}(B) = \mathsf{span}(P)$. Let N be the lowest common multiply of all the numbers N_B. By [16] the set $\mathsf{cone}(P)$ is the union of all the sets $\mathsf{cone}(B)$ where $B \subseteq P$ is linearly independent with $\mathsf{span}(B) = \mathsf{span}(P)$. It follows $\mathsf{L}(\boldsymbol{0}, P) \cap (N \cdot \mathbb{N}^k) = \mathsf{cone}(P) \cap (N \cdot \mathbb{N}^k)$.

For all $\nu, \xi \in \mathbb{N}$ it holds

$$\nu N \cdot \left(\sum_{i=1}^{|P|} \mu_i \boldsymbol{p_i} \right) + \xi N \cdot \sum_{i=1}^{|P|} (\lambda_i - \mu_i)\boldsymbol{p_i} \in \mathsf{L}(\boldsymbol{0}, P)$$

if and only if $\nu \geq \xi \cdot (R + 1)$. So, for $\nu_1, \nu_2 \in \mathbb{N}$ with $\nu_1 \neq \nu_2$ we have

$$\left[\nu_1 N \cdot \left(\sum_{i=1}^{|P|} \mu_i \boldsymbol{p_i} \right) \right]_{\mathsf{L}(\boldsymbol{0},P)} \neq \left[\nu_2 N \cdot \left(\sum_{i=1}^{|P|} \mu_i \boldsymbol{p_i} \right) \right]_{\mathsf{L}(\boldsymbol{0},P)},$$

which implies $\left|\mathsf{L}(\boldsymbol{0}, P)/\equiv_{\mathsf{L}(\boldsymbol{0},P)}\right| = \infty$. This proves the theorem. □

For linear sets with linearly independent periods we even get a stronger equivalence than in Theorem 6:

Corollary 7. *For $k \geq 0$ and a linearly independent $P \subseteq \mathbb{N}^k$ the following three conditions are equivalent:*

1. $\left|\mathsf{L}(\boldsymbol{0}, P)/\equiv_{\mathsf{L}(\boldsymbol{0},P)}\right| < \infty$
2. $\left|\mathsf{L}(\boldsymbol{0}, P)/\equiv_{\mathsf{L}(\boldsymbol{0},P)}\right| = 1$
3. *The set P has the own rational cone property.*

Proof. Let $P = \{\boldsymbol{p_1}, \boldsymbol{p_2}, \ldots, \boldsymbol{p_{|P|}}\}$ and assume that P has the own rational cone property. Let $\boldsymbol{x} \in \mathsf{L}(\boldsymbol{0}, P)$ and $\boldsymbol{y} \in \mathbb{N}^k$; otherwise, if $\boldsymbol{y} \notin \mathsf{span}(P)$, we get $\boldsymbol{x} + \boldsymbol{y} \notin \mathsf{L}(\boldsymbol{0}, P)$. If $\boldsymbol{y} \in \mathsf{span}(P)$, then we have $\boldsymbol{y} \in \mathsf{cone}(P)$, because P has the own independent rational cone property. So, there are uniquely determined values $\lambda_1, \lambda_2, \ldots, \lambda_{|P|} \in \mathbb{R}_{\geq 0}$ with $\boldsymbol{y} = \sum_{i=1}^{|P|} \lambda_i \boldsymbol{p_i}$. It holds $\boldsymbol{x} + \boldsymbol{y} \in \mathsf{L}(\boldsymbol{0}, P)$ if and only if for each $i \in \{1, 2, \ldots, |P|\}$ we have $\lambda_i \in \mathbb{N}$. This shows $\left|\mathsf{L}(\boldsymbol{0}, P)/\equiv_{\mathsf{L}(\boldsymbol{0},P)}\right| = 1$. Together with Theorem 6, that proves the corollary. □

3.2 Decompositions of Lattices

Lattices defined as subsets of $\mathbb{R}^k$ play an important rule in geometry, group theory, and cryptography, see [6]. Our lattices defined as subsets of $\mathbb{N}^k$ are a natural way to extend "the pattern" of a linear set to $\mathbb{N}^k$. Using lattices we can give a characterization in which cases arbitrary subsets of $\mathbb{N}^k$ can be decomposed as a finite union of subsets with only a finite number of positive Myhill-Nerode classes in the next subsection. This result, in turn, will enable us to prove a characterization result about MROWJFAs in the next section. In this subsection, we will show some decomposition results about lattices: it will turn out that lattices can be decomposed as a finite union of linear sets which have only one positive Myhill-Nerode equivalence class. Since each semi-linear set is the finite union of linear sets with linearly independent period sets by Theorem 2, we will investigate in which cases lattices can even be decomposed as a finite union of linear sets that have linearly independent period sets and only one positive Myhill-Nerode equivalence class (or only a finite number of positive Myhill-Nerode equivalence classes).

For $k \geq 0$, $\boldsymbol{c}, \boldsymbol{y} \in \mathbb{N}^k$, a finite $P \subseteq \mathbb{N}^k$, and $\boldsymbol{x} \in \mathsf{La}(\boldsymbol{c}, P)$ the vector $\boldsymbol{x} + \boldsymbol{y}$ is in $\mathsf{La}(\boldsymbol{c}, P)$ if and only if $\boldsymbol{y} \in \mathsf{La}(\boldsymbol{0}, P)$. This gives us that each lattice has only one positive Myhill-Nerode equivalence class. On the other hand, each lattice is a finite union of linear sets that have only one positive Myhill-Nerode equivalence class:

Proposition 8. *Let $k \geq 0$, $\boldsymbol{c} \in \mathbb{N}^k$, and $P \subseteq \mathbb{N}^k$ be finite. Then, there is a number $m > 0$, vectors $\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_m} \in \mathbb{N}^k$, and a finite $Q \subseteq \mathbb{N}^k$ such that $\mathsf{La}(\boldsymbol{c}, P) = \bigcup_{i=1}^{m} \mathsf{L}(\boldsymbol{c_i}, Q)$ and $\left|\mathsf{L}(\boldsymbol{0}, Q)/ \equiv_{\mathsf{L}(\boldsymbol{0},Q)}\right| = 1$.*

Proof. Consider the $\leq$-relation on $\mathbb{N}^k$ and let $M_{\boldsymbol{c}}$ (M, respectively) be the set of minimal elements of $\mathsf{La}(\boldsymbol{c}, P)$ ($\mathsf{La}(\boldsymbol{0}, P) \setminus \{\boldsymbol{0}\}$, respectively). Due to [7] each subset of $\mathbb{N}^k$ has only a finite number of minimal elements, so $M_{\boldsymbol{c}}$ and M are finite. It holds $\bigcup_{\boldsymbol{x} \in M_{\boldsymbol{c}}} \mathsf{L}(\boldsymbol{x}, M) \subseteq \mathsf{La}(\boldsymbol{c}, P)$.

We will show via induction over n that for all $n \geq 0$ we have

$$\{\, \boldsymbol{v} \in \mathsf{La}(\boldsymbol{c}, P) \mid ||\boldsymbol{v}||_1 \leq n \,\} \subseteq \bigcup_{\boldsymbol{x} \in M_{\boldsymbol{c}}} \mathsf{L}(\boldsymbol{x}, M). \tag{1}$$

This is clearly true for $n = 0$. So let now $n > 0$ and assume that there is an element $\boldsymbol{v} \in \mathsf{La}(\boldsymbol{c}, P)$ with $||\boldsymbol{v}||_1 \leq n$ and $\boldsymbol{v} \notin M_{\boldsymbol{c}}$. Then, there is a $\boldsymbol{y} \in M_{\boldsymbol{c}}$ with $\boldsymbol{y} \leq \boldsymbol{v}$ and a $\boldsymbol{w} \in M$ with $\boldsymbol{w} \leq \boldsymbol{v} - \boldsymbol{y}$. By the induction hypothesis we have $\boldsymbol{v} - \boldsymbol{w} \in \bigcup_{\boldsymbol{x} \in M_{\boldsymbol{c}}} \mathsf{L}(\boldsymbol{x}, M)$. It follows $\boldsymbol{v} = (\boldsymbol{v} - \boldsymbol{w}) + \boldsymbol{w} \in \bigcup_{\boldsymbol{x} \in M_{\boldsymbol{c}}} \mathsf{L}(\boldsymbol{x}, M)$. So relation (1) holds for all $n \geq 0$, which implies $\mathsf{La}(\boldsymbol{c}, P) = \bigcup_{\boldsymbol{x} \in M_{\boldsymbol{c}}} \mathsf{L}(\boldsymbol{x}, M)$.

We have $\mathsf{La}(\boldsymbol{0}, P) = \bigcup_{\boldsymbol{x} \in M_{\boldsymbol{0}}} \mathsf{L}(\boldsymbol{x}, M) = \mathsf{L}(\boldsymbol{0}, M)$. Now the proposition follows from the fact that each lattice has only one positive Myhill-Nerode class. $\square$

Next, we want to turn to the question, given a lattice as a finite union of linear sets, how the period set of the lattice is related to the period sets of the linear sets. To answer this question, we need the next result.

Lemma 9. *Let $k > 0$, $m \geq 0$, $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_m} \in \mathbb{R}^k$, and $V_1, V_2, \ldots, V_m \subset \mathbb{R}^k$ be proper linear subspaces of $\mathbb{R}^k$. Furthermore, let $r \in \mathbb{R}_{\geq 0}$. Then, there is a vector $\boldsymbol{w} \in \mathbb{R}^k$ with $B_r(\boldsymbol{w}) \subset (\mathbb{R}_{\geq 0})^k \setminus \bigcup_{i=1}^{m} (V_i + \boldsymbol{v_i})$.*

Proof. We prove this by induction on m. The statement is clearly true for $m = 0$. For $m > 0$ there is an $\boldsymbol{x} \in \mathbb{R}^k$ with $B_{3r+2}(\boldsymbol{x}) \subset (\mathbb{R}_{\geq 0})^k \setminus \bigcup_{i=1}^{m-1} (V_i + \boldsymbol{v_i})$ by the induction hypothesis. Let $V_m^{\perp}$ be the orthogonal complement of V_m. There are uniquely determined vectors $\boldsymbol{y_1} \in V_m$ and $\boldsymbol{y_2} \in V_m^{\perp}$ with $\boldsymbol{x} - \boldsymbol{v_m} = \boldsymbol{y_1} + \boldsymbol{y_2}$. For each $\boldsymbol{z} \in V_m$ we have

$$||\boldsymbol{z} + \boldsymbol{v_m} - \boldsymbol{x}||_2 = ||\boldsymbol{z} - \boldsymbol{y_1} - \boldsymbol{y_2}||_2 = \sqrt{||\boldsymbol{z} - \boldsymbol{y_1}||_2^2 + ||\boldsymbol{y_2}||_2^2} \geq ||\boldsymbol{y_2}||_2$$

by the Pythagorean theorem. If $||\boldsymbol{y_2}||_2 > r$ it follows

$$B_r(\boldsymbol{x}) \subset \left((\mathbb{R}_{\geq 0})^k \setminus \bigcup_{i=1}^{m-1} (V_i + \boldsymbol{v_i}) \right) \setminus (V_m + \boldsymbol{v_m}) = (\mathbb{R}_{\geq 0})^k \setminus \bigcup_{i=1}^{m} (V_i + \boldsymbol{v_i}).$$

Let now $||\boldsymbol{y_2}||_2 \leq r$. Then, let $\boldsymbol{y_3} \in V_m^\perp$ be arbitrary with $||\boldsymbol{y_3}||_2 = r + 1$. For each $\boldsymbol{z} \in V_m$ we have

$$||\boldsymbol{z} + \boldsymbol{v_m} - (\boldsymbol{y_1} + \boldsymbol{y_3} + \boldsymbol{v_m})||_2 = ||\boldsymbol{z} - \boldsymbol{y_1} - \boldsymbol{y_3}||_2 \geq ||\boldsymbol{y_3}||_2 = r + 1$$

again by the Pythagorean theorem. For each $\boldsymbol{u} \in \mathbb{R}^k$ with $\boldsymbol{u} \notin (\mathbb{R}_{\geq 0})^k \setminus \bigcup_{i=1}^{m-1} (V_i + \boldsymbol{v_i})$ it holds

$$\begin{aligned} ||\boldsymbol{u} - (\boldsymbol{y_1} + \boldsymbol{y_3} + \boldsymbol{v_m})||_2 &= ||\boldsymbol{u} - \boldsymbol{x} + \boldsymbol{y_2} - \boldsymbol{y_3}||_2 \geq ||\boldsymbol{u} - \boldsymbol{x}||_2 - ||\boldsymbol{y_3} - \boldsymbol{y_2}||_2 \\ &\geq 3r + 2 - (||\boldsymbol{y_2}||_2 + ||\boldsymbol{y_3}||_2) \geq 3r + 2 - (2r + 1) = r + 1. \end{aligned}$$

This implies

$$\begin{aligned} B_r(\boldsymbol{y_1} + \boldsymbol{y_3} + \boldsymbol{v_m}) &\subset \left((\mathbb{R}_{\geq 0})^k \setminus \bigcup_{i=1}^{m-1} (V_i + \boldsymbol{v_i}) \right) \setminus (V_m + \boldsymbol{v_m}) \\ &= (\mathbb{R}_{\geq 0})^k \setminus \bigcup_{i=1}^{m} (V_i + \boldsymbol{v_i}), \end{aligned}$$

which proves the lemma. $\square$

We can generalize Lemma 9:

Corollary 10. *Let $k > 0$, $\boldsymbol{v} \in (\mathbb{R}_{\geq 0})^k$, and V be a linear subspace of $\mathbb{R}^k$ that has a basis which is a subset of $(\mathbb{R}_{\geq 0})^k$. Moreover, let $m \geq 0$, $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_m} \in V + \boldsymbol{v}$, and $V_1, V_2, \ldots, V_m \subset V$ be proper linear subspaces of V. Furthermore, let $r \in \mathbb{R}_{\geq 0}$. Then, there is a $\boldsymbol{w} \in V + \boldsymbol{v}$ with $B_r(\boldsymbol{w}) \cap (V + \boldsymbol{v}) \subset (\mathbb{R}_{\geq 0})^k \setminus \bigcup_{i=1}^{m} (V_i + \boldsymbol{v_i})$.*

Proof. The statement is clearly true for $V = \{0\}$, so assume $\{0\} \subset V$ from now on. Let $n = \dim(V)$, $\{\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_n}\} \subset (\mathbb{R}_{\geq 0})^k$ be a basis of V, and $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_n}\}$ be the standard basis of $\mathbb{R}^n$. Consider the $\mathbb{R}$-linear map

$$f : \mathbb{R}^n \to \mathbb{R}^k \quad \text{defined by} \quad (\lambda_1, \lambda_2, \ldots, \lambda_n) \mapsto \sum_{i=1}^{n} \lambda_i \boldsymbol{b_i},$$

which is injective and continuous. Set

$$q = \min \{ \, ||f(\boldsymbol{x})||_2 \mid \boldsymbol{x} \in \mathbb{R}^n, \, ||\boldsymbol{x}||_2 = 1 \, \} > 0.$$

The map $g : \mathbb{R}^n \to V$ defined by $\boldsymbol{x} \mapsto f(\boldsymbol{x})/q$ is an isomorphism of vector spaces. Therefore $h : \mathbb{R}^n \to V + \boldsymbol{v}$ with $\boldsymbol{x} \mapsto g(\boldsymbol{x}) + \boldsymbol{v}$ is a bijection, with $h^{-1}(\boldsymbol{y}) = g^{-1}(\boldsymbol{y} - \boldsymbol{v})$ for all $\boldsymbol{y} \in V + \boldsymbol{v}$. By Lemma 9 there is a $\boldsymbol{w'} \in \mathbb{R}^n$ with

$$\begin{aligned} B_r(\boldsymbol{w'}) \subset (\mathbb{R}_{\geq 0})^n \setminus \bigcup_{i=1}^{m} \left(g^{-1}(V_i) + h^{-1}(\boldsymbol{v_i}) \right) &= (\mathbb{R}_{\geq 0})^n \setminus \bigcup_{i=1}^{m} g^{-1}(V_i + \boldsymbol{v_i} - \boldsymbol{v}) \\ &= (\mathbb{R}_{\geq 0})^n \setminus \bigcup_{i=1}^{m} h^{-1}(V_i + \boldsymbol{v_i}). \end{aligned}$$

For all $\boldsymbol{x}, \boldsymbol{z} \in \mathbb{R}^n$ with $\boldsymbol{x} \neq \boldsymbol{z}$ we have

$$||h(\boldsymbol{x}) - h(\boldsymbol{z})||_2 = ||g(\boldsymbol{x} - \boldsymbol{z})||_2 = \frac{||\boldsymbol{x} - \boldsymbol{z}||_2}{q} \cdot \left|\left| f\left(\frac{\boldsymbol{x} - \boldsymbol{z}}{||\boldsymbol{x} - \boldsymbol{z}||_2}\right)\right|\right|_2 \geq ||\boldsymbol{x} - \boldsymbol{z}||_2.$$

This gives us

$$\begin{aligned} B_r(h(\boldsymbol{w'})) \cap (V + \boldsymbol{v}) \subseteq h\left(B_r(\boldsymbol{w'})\right) \subset h\left((\mathbb{R}_{\geq 0})^n \setminus \bigcup_{i=1}^{m} h^{-1}(V_i + \boldsymbol{v_i})\right) \\ \subseteq (\mathbb{R}_{\geq 0})^k \setminus \bigcup_{i=1}^{m} (V_i + \boldsymbol{v_i}), \end{aligned}$$

which proves the corollary. □

Using Corollary 10, we can show that whenever we write a lattice as a finite union of linear sets, then at least one of the period sets of the linear sets spans the same vector space as the period set of the lattice:

Corollary 11. *Let $k \geq 0$, $\boldsymbol{c} \in \mathbb{N}^k$, and $P \subseteq \mathbb{N}^k$ be a finite set of vectors. Furthermore, let there be an $m > 0$, vectors $\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_m} \in \mathbb{N}^k$, and finite subsets $Q_1, Q_2, \ldots, Q_m \subseteq \mathbb{N}^k$ such that $\mathsf{La}(\boldsymbol{c}, P) = \bigcup_{i=1}^{m} \mathsf{L}(\boldsymbol{c_i}, Q_i)$. Then, it holds $\bigcup_{i=1}^{m} Q_i \subseteq \mathsf{span}(P)$ and there is at least one $i \in \{1, 2, \ldots, m\}$ with $\mathsf{span}(Q_i) = \mathsf{span}(P)$.*

Proof. The statement is clearly true for $k = 0$, so let $k > 0$. For each $i \in \{1, 2, \ldots, m\}$ we have $\boldsymbol{c_i} \in \mathsf{La}(\boldsymbol{c}, P)$ which implies $\boldsymbol{c_i} - \boldsymbol{c} \in \mathsf{span}(P)$. For each $i \in \{1, 2, \ldots, m\}$ and $\boldsymbol{p} \in Q_i$ it holds $\boldsymbol{p} = ((\boldsymbol{c_i} + \boldsymbol{p}) - \boldsymbol{c}) - (\boldsymbol{c_i} - \boldsymbol{c}) \in \mathsf{span}(P)$. That gives us $\bigcup_{i=1}^{m} Q_i \subseteq \mathsf{span}(P)$.

Assume now that $\mathsf{span}(Q_i) \subset \mathsf{span}(P)$, for all $i \in \{1, 2, \ldots, m\}$. Let $n = |P|$, where the set $P = \{\boldsymbol{p_1}, \boldsymbol{p_2}, \ldots, \boldsymbol{p_n}\}$, and $r = \left|\left|\sum_{j=1}^{n} \boldsymbol{p_j}\right|\right|_2$. By Corollary 10 there is a $\boldsymbol{w} \in \mathsf{span}(P) + \boldsymbol{c}$ with

$$B_r(\boldsymbol{w}) \cap (\mathsf{span}(P) + \boldsymbol{c}) \subset (\mathbb{R}_{\geq 0})^k \setminus \bigcup_{i=1}^{m} (\mathsf{span}(Q_i) + \boldsymbol{c_i}).$$

There are $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ with $\boldsymbol{w} = \boldsymbol{c} + \sum_{j=1}^{n} \lambda_j \boldsymbol{p_j}$. Set

$$\boldsymbol{v} = \boldsymbol{c} + \sum_{j=1}^{n} \lceil \lambda_j \rceil \cdot \boldsymbol{p_j} \in \mathsf{La}(\boldsymbol{c}, P).$$

From $||\boldsymbol{v} - \boldsymbol{w}||_2 \leq r$ we get $\boldsymbol{v} \notin \bigcup_{i=1}^{m} (\mathsf{span}(Q_i) + \boldsymbol{c_i}) \supseteq \bigcup_{i=1}^{m} \mathsf{L}(\boldsymbol{c_i}, Q_i)$. This contradiction proves the corollary. □

Now we are ready to show how the linearly independent rational cone property is connected to the property of lattices to be a finite union of linear sets that have linearly independent period sets and only finitely many positive Myhill-Nerode equivalence classes:

Theorem 12. *For $k \geq 0$, $\boldsymbol{c} \in \mathbb{N}^k$, and a finite $P \subseteq \mathbb{N}^k$ the following three conditions are equivalent:*

1. *There is an $m > 0$, vectors $\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_m} \in \mathbb{N}^k$, and linearly independent $Q_1, Q_2, \ldots, Q_m \subseteq \mathbb{N}^k$ such that $\mathsf{La}(\boldsymbol{c}, P) = \bigcup_{i=1}^{m} \mathsf{L}(\boldsymbol{c_i}, Q_i)$ and $\left|\mathsf{L}(\boldsymbol{0}, Q_i)/\equiv_{\mathsf{L}(\boldsymbol{0}, Q_i)}\right| < \infty$, for all $i \in \{1, 2, \ldots, m\}$.*
2. *There is an $m > 0$, vectors $\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_m} \in \mathbb{N}^k$, and a linearly independent $Q \subseteq \mathbb{N}^k$ such that $\mathsf{La}(\boldsymbol{c}, P) = \bigcup_{i=1}^{m} \mathsf{L}(\boldsymbol{c_i}, Q)$ and $\left|\mathsf{L}(\boldsymbol{0}, Q)/\equiv_{\mathsf{L}(\boldsymbol{0}, Q)}\right| = 1$.*

3. The set P has the linearly independent rational cone property.

Proof. Assume that P does not have the linearly independent rational cone property and let $m > 0$, $\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_m} \in \mathbb{N}^k$, and $Q_1, Q_2, \ldots, Q_m \subseteq \mathbb{N}^k$ be linearly independent subsets with $\mathsf{La}(\boldsymbol{c}, P) = \bigcup_{i=1}^m \mathsf{L}(\boldsymbol{c_i}, Q_i)$. By Corollary 11 there is an $i \in \{1, 2, \ldots, m\}$ with $\mathsf{span}(Q_i) = \mathsf{span}(P)$. So, the set Q_i does not have the linearly independent rational cone property, which implies that Q_i does not have the own rational cone property. By Corollary 7 we get $\left|\mathsf{L}(\boldsymbol{0}, Q_i)/\equiv_{\mathsf{L}(\boldsymbol{0},Q_i)}\right| = \infty$.

Assume now that P has the linearly independent rational cone property and let $T \subseteq \mathbb{N}^k$ be linearly independent with $\mathsf{span}(P) \cap (\mathbb{R}_{\geq 0})^k = \mathsf{cone}(T)$. We get $\mathsf{span}(P) = \mathsf{span}(T)$ and that T has the own rational cone property. Let $n = |T|$ and $T = \{\boldsymbol{t_1}, \boldsymbol{t_2}, \ldots, \boldsymbol{t_n}\}$. We get $T \subseteq \{\sum_{\boldsymbol{x_i} \in P} \lambda_i \cdot \boldsymbol{x_i} \mid \lambda_i \in \mathbb{Q}\}$ by basic linear algebra, so for each $i \in \{1, 2, \ldots, n\}$ there are a $\mu_i \in \mathbb{N} \setminus \{0\}$ and a $\boldsymbol{q_i} \in \mathsf{La}(\boldsymbol{0}, P)$ with $\boldsymbol{q_i} = \mu_i \boldsymbol{t_i}$. Let $Q = \{\boldsymbol{q_1}, \boldsymbol{q_2}, \ldots, \boldsymbol{q_n}\}$. Since $\mathsf{span}(Q) = \mathsf{span}(T)$ and $\mathsf{cone}(Q) = \mathsf{cone}(T)$, the linearly independent set Q has the own rational cone property, so Corollary 7 gives us $\left|\mathsf{L}(\boldsymbol{0}, Q)/\equiv_{\mathsf{L}(\boldsymbol{0},Q)}\right| = 1$.

Let E be the finite set

$$E = \left\{ \sum_{i=1}^n \nu_i \boldsymbol{q_i} \;\middle|\; \nu_i \in \mathbb{R},\, 0 \leq \nu_i < 1 \right\} \cap \mathsf{La}(\boldsymbol{0}, P).$$

Consider the $\leq$-relation on $\mathbb{N}^k$ and let M be the set of minimal elements of $\mathsf{La}(\boldsymbol{c}, P)$. Due to [7] each subset of $\mathbb{N}^k$ has only a finite number of minimal elements, so M is finite. Let

$$S = \bigcup_{\boldsymbol{m} \in M,\, \boldsymbol{e} \in E} \mathsf{L}(\boldsymbol{m} + \boldsymbol{e}, Q).$$

Because of $M \subseteq \mathsf{La}(\boldsymbol{c}, P)$ and $E, Q \subseteq \mathsf{La}(\boldsymbol{0}, P)$, we have $S \subseteq \mathsf{La}(\boldsymbol{c}, P)$. In the following we will show $\mathsf{La}(\boldsymbol{c}, P) = S$.

Let $\boldsymbol{x} \in \mathsf{La}(\boldsymbol{c}, P)$. There is an $\boldsymbol{m} \in M$ with $\boldsymbol{m} \leq \boldsymbol{x}$. Because of $\boldsymbol{x} - \boldsymbol{m} \in \mathsf{La}(\boldsymbol{0}, P) \subseteq \mathsf{cone}(Q)$ there are $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}$ with $\boldsymbol{x} - \boldsymbol{m} = \sum_{i=1}^n \lambda_i \boldsymbol{q_i}$. We have

$$\sum_{i=1}^n \lfloor \lambda_i \rfloor \boldsymbol{q_i} \in \mathsf{L}(\boldsymbol{0}, Q) \subseteq \mathsf{La}(\boldsymbol{0}, P) \quad \text{and} \quad \boldsymbol{x} - \boldsymbol{m} - \sum_{i=1}^n \lfloor \lambda_i \rfloor \boldsymbol{q_i} = \sum_{i=1}^n (\lambda_i - \lfloor \lambda_i \rfloor) \boldsymbol{q_i} \in E.$$

It follows $\boldsymbol{x} \in S$. So we have shown $\mathsf{La}(\boldsymbol{c}, P) = S$, which proves the theorem. □

Because of Theorem 12 it is worthwhile to investigate the linearly independent rational cone property more. Intuitively one might think that this property always holds, but it turns out that this is only the case in dimension $k \leq 3$:

Theorem 13. *Let $k \geq 0$ and $n \in \{0, 1, \ldots, k\}$. Then, the condition that each $S \subseteq \mathbb{N}^k$ with $\dim(\mathsf{span}(S)) = n$ has the linearly independent rational cone property holds if and only if $n \in \{0, 1, 2, k\}$.*

Proof. An $S \subseteq \mathbb{N}^k$ with $\dim(\mathsf{span}(S)) = 0$ clearly has linearly independent rational cone property. For $S \subseteq \mathbb{N}^k$ with $\dim(\mathsf{span}(S)) = 1$ and a vector $\boldsymbol{v} \in S \setminus \{\boldsymbol{0}\}$ we have

$$\mathsf{span}(S) \cap (\mathbb{R}_{\geq 0})^k = \mathsf{cone}(\{v\}).$$

For $S \subseteq \mathbb{N}^k$ with $\dim(\mathsf{span}(S)) = k$ it holds $\mathsf{span}(S) = \mathbb{R}^k$ and $\mathsf{span}(S) \cap (\mathbb{R}_{\geq 0})^k = \mathsf{cone}(B)$, where B is the standard basis of $\mathbb{R}^k$.

Let now $S \subseteq \mathbb{N}^k$ with $\dim(\mathsf{span}(S)) = 2 < k$ and let $\{\boldsymbol{v}, \boldsymbol{w}\} \subseteq S$ be a basis of $\mathsf{span}(S)$. Let

$$\lambda = \min \{ \boldsymbol{v}_j / \boldsymbol{w}_j \mid j \in \{1, 2, \ldots, k\},\, \boldsymbol{w}_j > 0 \}$$

and set $\boldsymbol{v'} := \boldsymbol{v} - \lambda \boldsymbol{w}$. So $\{\boldsymbol{v'}, \boldsymbol{w}\}$ is a basis of $\mathsf{span}(S)$. For each $j \in \{1, 2, \ldots, k\}$ with $\boldsymbol{w}_j > 0$ we get

$$\boldsymbol{v'}_j = \boldsymbol{v}_j - \lambda \boldsymbol{w}_j \geq \boldsymbol{v}_j - \boldsymbol{v}_j / \boldsymbol{w}_j \cdot \boldsymbol{w}_j = 0.$$

This gives us $\boldsymbol{v'} \in (\mathbb{Q}_{\geq 0})^k$. Now let

$$\mu = \min \left\{ \boldsymbol{w}_j / \boldsymbol{v'}_j \;\middle|\; j \in \{1, 2, \ldots, k\},\, \boldsymbol{v'}_j > 0 \right\}$$

and set $\boldsymbol{w'} := \boldsymbol{w} - \mu \boldsymbol{v'}$. Thus $\{\boldsymbol{v'}, \boldsymbol{w'}\}$ is a basis of $\mathsf{span}(S)$. For each $j \in \{1, 2, \ldots, k\}$ with $\boldsymbol{v'}_j > 0$ it holds

$$\boldsymbol{w'}_j = \boldsymbol{w}_j - \mu \boldsymbol{v'}_j \geq \boldsymbol{w}_j - \boldsymbol{w}_j / \boldsymbol{v'}_j \cdot \boldsymbol{v'}_j = 0.$$

That implies $\boldsymbol{w'} \in (\mathbb{Q}_{\geq 0})^k$.

For all $j \in \{1, 2, \ldots, k\}$ with $\boldsymbol{w}_j > 0$ and $\boldsymbol{v}_j / \boldsymbol{w}_j = \lambda$ we have $\boldsymbol{v'}_j = 0$ and $\boldsymbol{w'}_j > 0$. For all $j \in \{1, 2, \ldots, k\}$ with $\boldsymbol{v'}_j > 0$ and $\boldsymbol{w}_j / \boldsymbol{v'}_j = \mu$ we have $\boldsymbol{w'}_j = 0$. Let $\nu, \xi \in \mathbb{N}$ with $\nu, \xi > 0$ and $\nu \boldsymbol{v'}, \xi \boldsymbol{w'} \in \mathbb{N}^k$. We get

$$\mathsf{span}(S) \cap (\mathbb{R}_{\geq 0})^k = \mathsf{span}(\{\nu \boldsymbol{v'}, \xi \boldsymbol{w'}\}) \cap (\mathbb{R}_{\geq 0})^k = \mathsf{cone}(\{\nu \boldsymbol{v'}, \xi \boldsymbol{w'}\}),$$

so S has the linearly independent rational cone property.

Let now $n \in \{3, 4, \ldots, k-1\}$ and consider the 4×4 matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}.$$

For each $p, q \geq 0$ let I_p be the $p \times p$ identity matrix, $J_{p,q}$ be the $p \times q$ all-ones matrix, and $0_{p,q}$ be the $p \times q$ zero matrix. Let $S_{k,n}$ be the $k \times (n+1)$ block matrix

$$S_{k,n} = \begin{pmatrix} A & J_{4,n-3} \\ J_{n-3,4} & J_{n-3,n-3} - I_{n-3} \\ 0_{k-n-1,4} & 0_{k-n-1,n-3} \end{pmatrix}.$$

We refer to the columns of $S_{k,n}$ as $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_{n+1}} \in \mathbb{N}^k$. It holds $\boldsymbol{v_1} = \boldsymbol{v_2} + \boldsymbol{v_3} - \boldsymbol{v_4}$. Let $\lambda_2, \lambda_3, \ldots, \lambda_{n+1} \in \mathbb{R}$, set $\boldsymbol{w} = \sum_{i=2}^{n+1} \lambda_i \cdot \boldsymbol{v_i}$, and assume $\boldsymbol{w} = \boldsymbol{0}$. Then we have $0 = \boldsymbol{w}_1 - \boldsymbol{w}_4 = 2\lambda_4$. For all $i \in \{5, 6, \ldots, n+1\}$ we have $0 = \boldsymbol{w}_4 - \boldsymbol{w}_i = \lambda_i$. Now it is easy to see that we also have $\lambda_2 = \lambda_3 = 0$. So the vectors $\boldsymbol{v_2}, \boldsymbol{v_3}, \ldots, \boldsymbol{v_{n+1}}$ are linearly independent and the matrix $S_{k,n}$ has rank n.

Let $V_{k,n}$ be the set of columns of $S_{k,n}$, assume that $V_{k,n}$ has the linearly independent rational cone property, and let $B = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_n}\} \subset \mathbb{N}^k$ be linearly independent with

$$\mathsf{span}(V_{k,n}) \cap (\mathbb{R}_{\geq 0})^k = \mathsf{cone}(B).$$

Furthermore, let $B = (b_{j,i})_{j \in \{1,2,\ldots,k\}, i \in \{1,2,\ldots,n\}}$ be the $k \times n$ matrix whose columns are from left to right $\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_n}$. We have $b_{j,i} = 0$ for all $j \in \{n+2, n+3, \ldots, k\}$ and $i \in \{1, 2, \ldots, n\}$.

Assume that there is an $i_0 \in \{1, 2, \ldots, n\}$ such that for all $j \in \{1, 2, \ldots, n+1\}$ there is an $i \in \{1, 2, \ldots, n\} \setminus \{i_0\}$ with $b_{j,i} > 0$. Set $\boldsymbol{x} = \sum_{i \in \{1,2,\ldots,n\} \setminus \{i_0\}} \boldsymbol{b_i}$. We have $\boldsymbol{x}_j > 0$ for all $j \in \{1, 2, \ldots, n+1\}$. Let

$$\lambda = \min \{ \boldsymbol{x}_j / b_{j,i_0} \mid j \in \{1, 2, \ldots, k\},\, b_{j,i_0} > 0 \} > 0$$

and set $\boldsymbol{x'} = \boldsymbol{x} - \lambda \boldsymbol{b_{i_0}}$. For each $j \in \{1, 2, \ldots, k\}$ with $b_{j,i_0} > 0$ we get

$$\boldsymbol{x'}_j = \boldsymbol{x}_j - \lambda b_{j,i_0} \geq \boldsymbol{x}_j - \boldsymbol{x}_j / b_{j,i_0} \cdot b_{j,i_0} = 0.$$

This is a contradiction to

$$\mathsf{span}(B) \cap (\mathbb{R}_{\geq 0})^k = \mathsf{span}(V_{k,n}) \cap (\mathbb{R}_{\geq 0})^k = \mathsf{cone}(B),$$

because $\boldsymbol{x'} = \sum_{i \in \{1,2,\ldots,n\} \setminus \{i_0\}} \boldsymbol{b_i} - \lambda \boldsymbol{b_{i_0}}$. So for all $i \in \{1, 2, \ldots, n\}$ there is a $j_i \in \{1, 2, \ldots, n+1\}$ such that for all $i' \in \{1, 2, \ldots, n\} \setminus \{i\}$ we have $b_{j_i,i'} = 0$. It follows $b_{j_i,i} > 0$ for all $i \in \{1, 2, \ldots, n\}$. Let

$$C = \{ j_i \mid i \in \{1, 2, \ldots, n\} \} \subset \{1, 2, \ldots, n+1\}.$$

We get $|C| = n$. Let m be the only element of $\{1, 2, \ldots, n+1\} \setminus C$. From what we have shown about the matrix B it follows that for every $\boldsymbol{y} \in \mathsf{span}(B)$ it holds: if $\boldsymbol{y}_j \geq 0$ for all $j \in C$, then we also have $\boldsymbol{y}_m \geq 0$.

Let now $i \in C$ and set $\boldsymbol{y} = \boldsymbol{v_m} - 1/3 \cdot \boldsymbol{v_i} \in \mathsf{span}(V_{k,n}) = \mathsf{span}(B)$. This gives us $\boldsymbol{y}_m < 0$. For all $j \in C$ we have $\boldsymbol{y}_j \geq 1 - 1/3 \cdot 2 = 1/3$, a contradiction. So, the set $V_{k,n}$ does not have the linearly independent rational cone property. This proves the theorem. ☐

Thus, for $k \geq 0$ and $n \in \{0, 1, \ldots, k\}$, the condition that for all vectors $\boldsymbol{c} \in \mathbb{N}^k$ and finite sets $P \subseteq \mathbb{N}^k$ with $\dim(\mathsf{span}(P)) = n$ we get a decomposition of the set $\mathsf{La}(\boldsymbol{c}, P)$ as in Theorem 12 is equivalent to the condition $n \in \{0, 1, 2, k\}$.

3.3 A Decomposition Result About Subsets of $\mathbb{N}^k$

Having the decompositions of lattices from Subsection 3.2, we now turn to a decomposition result about arbitrary subsets of $\mathbb{N}^k$. To state the result, we will work with quasi lattices. The are defined as follows: let $k \geq 0$ and $S \subseteq \mathbb{N}^k$. The set S is a *quasi lattice* if and only if there is a vector $\boldsymbol{y} \in \mathbb{N}^k$, an $m \geq 0$, vectors $\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_m} \in \mathbb{N}^k$, and finite subsets $P_1, P_2, \ldots, P_m \subseteq \mathbb{N}^k$ such that the set $\{ \boldsymbol{z} \in S \mid \boldsymbol{z} \geq \boldsymbol{y} \}$ is equal to $\{ \boldsymbol{z} \in \bigcup_{j=1}^m \mathsf{La}(\boldsymbol{c_j}, P_j) \mid \boldsymbol{z} \geq \boldsymbol{y} \}$.

We can identify a pattern of two linear sets formed by three vectors that gives a sufficient condition for the property of a subset of $\mathbb{N}^k$ to not be a quasi lattice:

Lemma 14. *Let $k \geq 0$ and $S \subseteq \mathbb{N}^k$ such that there are vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^k$ with $\pi_{k,\{j\}}(\boldsymbol{v}) > 0$, for all $j \in \{1, 2, \ldots, k\}$ with $\mathsf{L}(\boldsymbol{u}, \{\boldsymbol{v}\}) \cap S = \emptyset$ and moreover $\mathsf{L}(\boldsymbol{u} + \boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) \subseteq S$. Then, the set S is* not *a quasi lattice.*

Proof. Let vector $\boldsymbol{y} \in \mathbb{N}^k$. There is a $\lambda \in \mathbb{N}$ with $\boldsymbol{u} + \lambda \boldsymbol{v} \geq \boldsymbol{y}$. Consider an $m > 0$, vectors $\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_m} \in \mathbb{N}^k$, and finite subsets $P_1, P_2, \ldots, P_m \subseteq \mathbb{N}^k$ such that it holds

$$\mathsf{L}(\boldsymbol{u} + \boldsymbol{w} + \lambda \boldsymbol{v}, \{\boldsymbol{w}\}) \subseteq \bigcup_{j=1}^{m} \mathsf{La}(\boldsymbol{c_j}, P_j).$$

W.l.o.g. there is an $n > 0$ with $n \leq m$ such that for all $j \in \{1, 2, \ldots, m\}$ the inequality

$$|\mathsf{L}(\boldsymbol{u} + \boldsymbol{w} + \lambda \boldsymbol{v}, \{\boldsymbol{w}\}) \cap \mathsf{La}(\boldsymbol{c_j}, P_j)| \geq 2$$

holds if and only if $j \leq n$. For all $j \in \{1, 2, \ldots, n\}$ let M_j be the minimum of

$$\{\, \mu_2 - \mu_1 \mid \mu_1, \mu_2 \in \mathbb{N}, 0 < \mu_1 < \mu_2, \forall i \in \{1, 2\} : \boldsymbol{u} + \lambda \boldsymbol{v} + \mu_i \boldsymbol{w} \in \mathsf{La}(\boldsymbol{c_j}, P_j) \,\}$$

and let M be the lowest common multiple of the M_j. For all $j \in \{1, 2, \ldots, n\}$ and $\nu \geq 0$ we have $\boldsymbol{u}+\lambda\boldsymbol{v} \in \mathsf{La}(\boldsymbol{c_j}, P_j)$ if and only if $\boldsymbol{u}+\lambda\boldsymbol{v}+\nu M\boldsymbol{w} \in \mathsf{La}(\boldsymbol{c_j}, P_j)$. It follows $\boldsymbol{u}+\lambda\boldsymbol{v} \in \bigcup_{j=1}^{n} \mathsf{La}(\boldsymbol{c_j}, P_j)$. Since $\mathsf{L}(\boldsymbol{u} + \boldsymbol{w} + \lambda\boldsymbol{v}, \{\boldsymbol{w}\}) \subseteq S$, but $\boldsymbol{u} + \lambda\boldsymbol{v} \notin S$, the set S is not a quasi lattice. $\square$

We call subsets of $\mathbb{N}^k$ that allow a pattern as in Lemma 14 anti-lattices. If they even allow a pattern that begins at the origin, we call them 0-anti-lattices: let $k \geq 0$ and $S \subseteq \mathbb{N}^k$. If there are vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^k$ with $\pi_{k,\{j\}}(\boldsymbol{v}) > 0$ for all $j \in \{1, 2, \ldots, k\}$ so that $\mathsf{L}(\boldsymbol{u}, \{\boldsymbol{v}\}) \cap S = \emptyset$ and $\mathsf{L}(\boldsymbol{u}+\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) \subseteq S$, the set S is called an *anti-lattice*. If there are $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^k$ with $\pi_{k,\{j\}}(\boldsymbol{v}) > 0$ for all $j \in \{1, 2, \ldots, k\}$ so that $\mathsf{L}(\boldsymbol{0}, \{\boldsymbol{v}\}) \cap S = \emptyset$ and $\mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) \subseteq S$, the set S is called a *0-anti-lattice*. The following property of anti-lattices will be used later on.

Lemma 15. *Let $k \geq 0$ and $S \subseteq \mathbb{N}^k$. Then, the set S is an anti-lattice if and only if there is an $\boldsymbol{x} \in \mathbb{N}^k$ such that $S + \boldsymbol{x}$ is a 0-anti-lattice.*

Proof. The lemma is clearly true for $k = 0$, so let $k > 0$. Assume that there is an $\boldsymbol{x} \in \mathbb{N}^k$ such that $S+\boldsymbol{x}$ is a 0-anti-lattice. Thus, there are $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^k$ with $\pi_{k,\{j\}}(\boldsymbol{v}) > 0$ for all $j \in \{1, 2, \ldots, k\}$ so that $\mathsf{L}(\boldsymbol{0}, \{\boldsymbol{v}\}) \cap (S + \boldsymbol{x}) = \emptyset$ and $\mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) \subseteq S + \boldsymbol{x}$. Let $\lambda \in \mathbb{N}$ with $\lambda\boldsymbol{v} \geq \boldsymbol{x}$. Then, we get $\mathsf{L}(\lambda\boldsymbol{v} - \boldsymbol{x}, \{\boldsymbol{v}\}) \cap S = \emptyset$ and $\mathsf{L}(\lambda\boldsymbol{v} - \boldsymbol{x} + \boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) \subseteq S$. So, the set S is an anti-lattice.

Assume now that S is an anti-lattice. There are vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^k$ with $\pi_{k,\{j\}}(\boldsymbol{v}) > 0$ for all $j \in \{1, 2, \ldots, k\}$ so that

$$\mathsf{L}(\boldsymbol{u}, \{\boldsymbol{v}\}) \cap S = \emptyset \quad \text{and} \quad \mathsf{L}(\boldsymbol{u} + \boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) \subseteq S.$$

Let $\boldsymbol{x} \in \mathbb{N}^k \setminus \{\boldsymbol{0}\}$ and $\lambda \in \mathbb{N}$ with $\boldsymbol{u} + \boldsymbol{x} = \lambda\boldsymbol{v}$. We get $\mathsf{L}(\boldsymbol{0}, \{\lambda\boldsymbol{v}\}) \cap (S + \boldsymbol{x}) = \emptyset$ and moreover $\mathsf{L}(\lambda\boldsymbol{v} + \boldsymbol{w}, \{\lambda\boldsymbol{v}, \lambda\boldsymbol{v} + \boldsymbol{w}\}) \subseteq (S + \boldsymbol{x})$. So, the set $S + \boldsymbol{x}$ is a 0-anti-lattice. $\square$

A semi-linear set is a quasi lattice if and only if it is not an anti-lattice:

Proposition 16. *Let $k \geq 0$ and $S \subseteq \mathbb{N}^k$ be a semi-linear set. Then, the set S is a quasi lattice if and only if S is not an anti-lattice.*

Proof. Assume that S is not a quasi lattice. As a semi-linear set S is a finite union of linear sets, so there is a $\boldsymbol{c} \in \mathbb{N}^k$ and a finite subset $P \subseteq \mathbb{N}^k$ such that $\mathsf{L}(\boldsymbol{c}, P) \subseteq S$ and for all $\boldsymbol{y} \in \mathbb{N}^k$ it holds $\{\, \boldsymbol{z} \in \mathsf{La}(\boldsymbol{c}, P) \mid \boldsymbol{z} \geq \boldsymbol{y} \,\} \nsubseteq S$. Let $P = \{\boldsymbol{p_1}, \boldsymbol{p_2}, \ldots, \boldsymbol{p_{|P|}}\}$. For $\boldsymbol{y} \in \mathbb{N}^k$ let $N_{\boldsymbol{y}}$ be the non-empty set

$$\left\{ \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{|P|} \end{pmatrix} \in \mathbb{N}^{|P|} \,\middle|\, \exists \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{|P|} \end{pmatrix} \in \mathbb{N}^{|P|} : \sum_{j=1}^{|P|} (\mu_j - \lambda_j)\boldsymbol{p_j} \in \left\{ \boldsymbol{z} \in \mathbb{N}^k \setminus S \,\middle|\, \boldsymbol{z} \geq \boldsymbol{y} \right\} \right\}$$

and let $M_{\boldsymbol{y}}$ be the set of minimal elements of $N_{\boldsymbol{y}}$ regarding the $\leq$-relation on $\mathbb{N}^{|P|}$. For all $(\lambda_1, \lambda_2, \ldots, \lambda_{|P|}) \in M_{\boldsymbol{y}}$ there are $(\mu_1, \mu_2, \ldots, \mu_{|P|}) \in \mathbb{N}^{|P|}$ and j_0 with $j_0 \in \{1, 2, \ldots, |P|\}$ such that $\sum_{j=1}^{|P|} (\mu_j - \lambda_j)\boldsymbol{p_j} \in \{\, \boldsymbol{z} \in \mathbb{N}^k \setminus S \mid \boldsymbol{z} \geq \boldsymbol{y} \,\}$ and $\alpha\boldsymbol{p_{j_0}} + \sum_{j=1}^{|P|} (\mu_j - \lambda_j)\boldsymbol{p_j} \in S$, for all $\alpha > 0$.

For $m \geq 0$ let $\boldsymbol{y_m}$ be the vector in $\mathbb{N}^k$ which fulfils $\pi_{k,\{j\}}(\boldsymbol{y_m}) = m$ for all $j \in \{1, 2, \ldots, k\}$. Let now j_0 be in the non-empty set

$$\bigcap_{m=0}^{\infty} \left\{ j \in \{1, 2, \ldots, |P|\} \,\middle|\, \exists \boldsymbol{x} \in \mathbb{N}^k \setminus S : (\boldsymbol{x} \geq \boldsymbol{y_m} \wedge \forall \alpha > 0 : \boldsymbol{x} + \alpha\boldsymbol{p_j} \in S) \right\}.$$

The set $T = \{\, \boldsymbol{x} \in \mathbb{N}^k \setminus S \mid \forall \alpha > 0 : \boldsymbol{x} + \alpha \boldsymbol{p_{j_0}} \in S \,\}$ is semi-linear, because a subset of $\mathbb{N}^k$ is semi-linear if and only if it is a Presburger set due to [11]. So, there is $\ell \geq 0$, $\boldsymbol{d_1}, \boldsymbol{d_2}, \ldots, \boldsymbol{d_\ell} \in \mathbb{N}^k$, and finite $Q_1, Q_2, \ldots, Q_\ell \subseteq \mathbb{N}^k$ such that $T = \bigcup_{h=1}^{\ell} \mathsf{L}(\boldsymbol{d_h}, Q_h)$. Since for each $\boldsymbol{y} \in \mathbb{N}^k$ there is an $\boldsymbol{x} \in T$ with $\boldsymbol{x} \geq \boldsymbol{y}$, there exists an $h_0 \in \{1, 2, \ldots, \ell\}$ such that $\pi_{k,\{j\}} \left(\sum_{\boldsymbol{q} \in Q_{h_0}} \boldsymbol{q} \right) > 0$ for all $j \in \{1, 2, \ldots, k\}$. We have $L \left(\boldsymbol{d_{h_0}}, \left\{ \sum_{\boldsymbol{q} \in Q_{h_0}} \boldsymbol{q} \right\} \right) \cap S = \emptyset$ and

$$L \left(\boldsymbol{d_{h_0}} + \boldsymbol{p_{j_0}}, \left\{ \boldsymbol{p_{j_0}}, \sum_{\boldsymbol{q} \in Q_{h_0}} \boldsymbol{q} \right\} \right) \subseteq S.$$

So, the set S is an anti-lattice. Our proposition follows with Lemma 14. □

It follows that each subset of $\mathbb{N}^k$ which has only a finite number of positive Myhill-Nerode equivalence classes is a quasi lattice:

Corollary 17. *For a $k \geq 0$ and a subset $S \subseteq \mathbb{N}^k$ with $|S/\equiv_S| < \infty$ the set S is a quasi lattice.*

Proof. Let $k \geq 0$, $S \subseteq \mathbb{N}^k$ and $|S/\equiv_S| < \infty$. Since every language in the family **ROWJ** is semi-linear, see [2], Theorem 4 implies that S is semi-linear. Assume that S is an anti-lattice. So, there are $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^k$ with $\pi_{k,\{j\}}(\boldsymbol{v}) > 0$ for all $j \in \{1, 2, \ldots, k\}$ such that $\mathsf{L}(\boldsymbol{u}, \{\boldsymbol{v}\}) \cap S = \emptyset$ and $\mathsf{L}(\boldsymbol{u} + \boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) \subseteq S$. Set $\lambda_n = \min\{\, \lambda \in \mathbb{N} \mid n\boldsymbol{w} \leq \lambda \boldsymbol{v} \,\}$, for every $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$ we have

$$(\boldsymbol{u} + n\boldsymbol{w}) + (\lambda_n \boldsymbol{v} - n\boldsymbol{w}) = \boldsymbol{u} + \lambda_n \boldsymbol{v} \notin S,$$

but for all $m \in \mathbb{N}$ with $m < n$ it holds

$$(\boldsymbol{u} + n\boldsymbol{w}) + (\lambda_m \boldsymbol{v} - m\boldsymbol{w}) = \boldsymbol{u} + (n - m)\boldsymbol{w} + \lambda_m \boldsymbol{v} \in S.$$

Thus, for $n, m \in \mathbb{N}$ with $n \neq m$ we have $[\boldsymbol{u} + n\boldsymbol{w}]_{\equiv_S} \neq [\boldsymbol{u} + m\boldsymbol{w}]_{\equiv_S}$, which gives us $|S/\equiv_S| = \infty$, a contradiction. Therefore, the set S is a quasi lattice by Proposition 16. □

Quasi lattices are related to the property of a subset $S \subseteq \mathbb{N}^k$ to be a finite union of subsets of $\mathbb{N}^k$ that have only a finite number of positive Myhill-Nerode equivalence classes, which holds exactly if S is a finite union of linear sets that have only one positive Myhill-Nerode equivalence class:

Theorem 18. *For a $k \geq 0$ and a subset $S \subseteq \mathbb{N}^k$ the following three conditions are equivalent:*

1. *There is an $m \geq 0$ and subsets $S_1, S_2, \ldots, S_m \subseteq \mathbb{N}^k$ such that $S = \bigcup_{j=1}^{m} S_j$ and for each $j \in \{1, 2, \ldots, m\}$ we have $\left| S_j / \equiv_{S_j} \right| < \infty$.*
2. *There is an $m \geq 0$ and linear sets $L_1, L_2, \ldots, L_m \subseteq \mathbb{N}^k$ such that $S = \bigcup_{j=1}^{m} L_j$ and for each $j \in \{1, 2, \ldots, m\}$ we have $\left| L_j / \equiv_{L_j} \right| = 1$.*
3. *The set $\pi_{k,\{1,2,\ldots,k\} \setminus T} \left(\{\, \boldsymbol{z} \in S \mid \pi_{k,T}(\boldsymbol{z}) = \boldsymbol{x} \,\} \right)$ is a quasi lattice, for all sets $T \subseteq \{1, 2, \ldots, k\}$ and vectors $\boldsymbol{x} \in \mathbb{N}^{|T|}$.*

Proof. The first condition of the theorem implies the third one, because of Corollary 17, the fact that for all $T \subseteq \{1, 2, \ldots, k\}$ and $\boldsymbol{x} \in \mathbb{N}^{|T|}$ we have $\left| S_{T,\boldsymbol{x}} / \equiv_{S_{T,\boldsymbol{x}}} \right| \leq |S/\equiv_S|$, where

$$S_{T,\boldsymbol{x}} := \pi_{k,\{1,2,\ldots,k\} \setminus T} \left(\{\, \boldsymbol{z} \in S \mid \pi_{k,T}(\boldsymbol{z}) = \boldsymbol{x} \,\} \right),$$

and the fact that quasi lattices are closed under the operation of union.

We now prove by induction on k that the third condition of our theorem implies the second one. This is true for $k = 0$, so let $k > 0$ and assume that the third condition holds. Choosing $T = \emptyset$ in this condition gives us that S is a quasi lattice. So, there exists a vector $\boldsymbol{y} \in \mathbb{N}^k$, a number $n \geq 0$, vectors $\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_n} \in \mathbb{N}^k$, and finite subsets $P_1, P_2, \ldots, P_n \subseteq \mathbb{N}^k$ with

$$\{\, \boldsymbol{z} \in S \mid \boldsymbol{z} \geq \boldsymbol{y} \,\} = \bigcup_{i=1}^{n} \{\, \boldsymbol{z} \in \mathsf{La}(\boldsymbol{c_i}, P_i) \mid \boldsymbol{z} \geq \boldsymbol{y} \,\}.$$

We get

$$S = \{\, \boldsymbol{z} \in S \mid \boldsymbol{z} \ngeq \boldsymbol{y} \,\} \cup \bigcup_{i=1}^{n} \{\, \boldsymbol{z} \in \mathsf{La}(\boldsymbol{c_i}, P_i) \mid \boldsymbol{z} \geq \boldsymbol{y} \,\},$$

where $\{\, \boldsymbol{z} \in S \mid \boldsymbol{z} \ngeq \boldsymbol{y} \,\} = \bigcup_{p=1}^{k} \bigcup_{q=0}^{\pi_{k,\{p\}}(\boldsymbol{y})-1} \{\, \boldsymbol{z} \in S \mid \pi_{k,\{p\}}(\boldsymbol{z}) = q \,\}$. Now let $p \in \{1, 2, \ldots, k\}$ and $q \in \{0, 1, \ldots, \pi_{k,\{p\}}(\boldsymbol{y}) - 1\}$ and define the map $\iota_{p,q} : \mathbb{N}^{k-1} \to \mathbb{N}^k$ as

$$\iota_{p,q}(\boldsymbol{x}) = (\pi_{k-1,\{1,2,\ldots,p-1\}}(\boldsymbol{x}), q, \pi_{k-1,\{p,p+1,\ldots,k-1\}}(\boldsymbol{x})).$$

By induction hypothesis there is a $m_{p,q} \geq 0$ and linear subsets $L_{p,q,1}, L_{p,q,2}, \ldots, L_{p,q,m_{p,q}} \subseteq \mathbb{N}^{k-1}$ such that

$$\iota_{p,q}^{-1}\left(\{\, \boldsymbol{z} \in S \;\middle|\; \pi_{k,\{p\}}(\boldsymbol{z}) = q \,\}\right) = \bigcup_{j=1}^{m_{p,q}} L_{p,q,j}$$

and for each $j \in \{1, 2, \ldots, m_{p,q}\}$ we have $\left| L_{p,q,j} / \equiv_{L_{p,q,j}} \right| = 1$. It follows

$$\{\, \boldsymbol{z} \in S \;\middle|\; \pi_{k,\{p\}}(\boldsymbol{z}) = q \,\} = \bigcup_{j=1}^{m_{p,q}} \iota_{p,q}(L_{p,q,j}),$$

where for each $j \in \{1, 2, \ldots, m_{p,q}\}$ the set $\iota_{p,q}(L_{p,q,j})$ is linear and $\left| \iota_{p,q}(L_{p,q,j}) / \equiv_{\iota_{p,q}(L_{p,q,j})} \right| = 1$.

By Proposition 8 for each i with $i \in \{1, 2, \ldots, n\}$ there is an $m_i > 0$ and linear subsets $L_{i,1}, L_{i,2}, \ldots, L_{i,m_i} \subseteq \mathbb{N}^k$ such that $\mathsf{La}(\boldsymbol{0}, P_i) = \bigcup_{j=1}^{m_i} L_{i,j}$ and for each $j \in \{1, 2, \ldots, m_i\}$ we have $\left| L_{i,j} / \equiv_{L_{i,j}} \right| = 1$. Let M_i be the set of minimal elements of $\{\, \boldsymbol{z} \in \mathsf{La}(\boldsymbol{c_i}, P_i) \mid \boldsymbol{z} \geq \boldsymbol{y} \,\}$ regarding the $\leq$-relation on $\mathbb{N}^k$. We get

$$\{\, \boldsymbol{z} \in \mathsf{La}(\boldsymbol{c_i}, P_i) \mid \boldsymbol{z} \geq \boldsymbol{y} \,\} = \bigcup_{\boldsymbol{x} \in M_i} (\mathsf{La}(\boldsymbol{0}, P_i) + \boldsymbol{x}) = \bigcup_{\boldsymbol{x} \in M_i} \bigcup_{j=1}^{m_i} (L_{i,j} + \boldsymbol{x}).$$

So, it holds

$$S = \left(\bigcup_{p=1}^{k} \bigcup_{q=0}^{\pi_{k,\{p\}}(\boldsymbol{y})-1} \bigcup_{j=1}^{m_{p,q}} \iota_{p,q}(L_{p,q,j}) \right) \cup \bigcup_{i=1}^{n} \bigcup_{\boldsymbol{x} \in M_i} \bigcup_{j=1}^{m_i} (L_{i,j} + \boldsymbol{x}).$$

So, the second condition of the theorem holds. $\square$

In dimension $k \leq 3$ we can strengthen the second condition of Theorem 18, while we can weaken the third condition in dimension $k \leq 2$:

Corollary 19. *For a $k \in \{0, 1, 2, 3\}$ and a subset $S \subseteq \mathbb{N}^k$ the conditions from Theorem 18 are equivalent to the following condition. There is a number $m \geq 0$, vectors $\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_m} \in \mathbb{N}^k$,* *and linearly independent sets* $P_1, P_2, \ldots, P_m \subseteq \mathbb{N}^k$ *such that it holds* $S = \bigcup_{j=1}^{m} \mathsf{L}(\boldsymbol{c_j}, P_j)$ *and for each* $j \in \{1, 2, \ldots, m\}$ *we have* $\left|\mathsf{L}(\mathbf{0}, P_j)/\equiv_{\mathsf{L}(\mathbf{0},P_j)}\right| = 1$*. For a* $k \in \{0, 1, 2\}$ *and a subset* $S \subseteq \mathbb{N}^k$ *the conditions from Theorem 18 are equivalent to the condition that* S *is a semi-linear set and a quasi lattice.*

Proof. We get the statement for $k \in \{0, 1, 2, 3\}$ by using Theorems 12 and 13 instead of Proposition 8 in the last paragraph of the proof of Theorem 18. The statement for $k \in \{0, 1, 2\}$ follows from the fact that a subset of $\mathbb{N}$ is a quasi lattice if and only if it is a semi-linear set. That holds by Theorem 2 and the fact that semi-linear sets are closed under the operation of set difference, see [9]. □

4 Right One-Way Jumping Finite Automata with Multiple Initial States

In this section we investigate MROWJFAs. To get deep results about these devices we use results from Subsection 3.3.

4.1 Results for Arbitrary Alphabets

First, some basic properties are given. Directly from the definition of MROWJFAs we get that the unary languages in **MROWJ** are exactly the unary regular languages and that **MROWJ** consists exactly of the finite unions of languages from **ROWJ**. However, it is not clear that every language from **pMROWJ** is a finite union of languages from **pROWJ**. From [2] we know that a^* and the language $\{\, w \in \{a, b\}^* \mid |w|_a = |w|_b \,\}$ are in **ROWJ**, but the union of these two sets is not in **ROWJ**. Together with the properties of **ROWJ** shown in [2] and [4], this gives us: we have **REG** $\subset$ **ROWJ** $\subset$ **MROWJ** and also **pREG** $\subset$ **pROWJ** $\subset$ **pMROWJ**. The family **MROWJ** in incomparable to **DCF** and to **CF**. Each language in **MROWJ** is semi-linear and contained in the complexity classes **DTIME**(n^2) and **DSPACE**(n). We get **pMROWJ** $\nsubseteq$ **CF** and **pMROWJ** $\subseteq$ **JFA** $\subset$ **pCS**. The letter-bounded languages contained in **MROWJ** are exactly the regular letter-bounded languages.

Now, we will study the language class **pMROWJ** in depth. The foundation for this will be the next result.

Theorem 20. *The family* **pMROWJ** *does not contain any language whose Parikh-image is a 0-anti-lattice.*

Proof. Assume that there is a $k > 0$, an $S \subseteq \mathbb{N}^k$, and an ordered alphabet Σ with $|\Sigma| = k$ such that S is a 0-anti-lattice, but the language $\psi^{-1}(S) \subseteq \Sigma^*$ is in **MROWJ**. So, there are $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^k$, where for all $j \in \{1, 2, \ldots, k\}$ it holds $\pi_{k,\{j\}}(\boldsymbol{v}) > 0$, such that we have $\mathsf{L}(\mathbf{0}, \{\boldsymbol{v}\}) \cap S = \emptyset$ and $\mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) \subseteq S$. Clearly we have $\boldsymbol{w} \neq \mathbf{0}$ and $\boldsymbol{v} \neq \mathbf{0}$. Since

$$\mathsf{L}(\mathbf{0}, \{\boldsymbol{v}\}) \cap \mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) = \emptyset,$$

there are no $\lambda, \mu \in \mathbb{N}$ with $\lambda, \mu > 0$ and $\lambda\boldsymbol{v} = \mu\boldsymbol{w}$, which implies that $\{\boldsymbol{v}, \boldsymbol{w}\}$ is linearly independent. Let $A = (Q, \Sigma, R, S', F)$ be an MROWJFA with $L_R(A) = \psi^{-1}(S)$ and assume that the set of initial states $S' = \{s_1, s_2, \ldots, s_{|S'|}\}$. For two states $p, q \in Q$ let $L_{p,q}$ be the language accepted by the ordinary DFA $(Q, \Sigma, R, p, \{q\})$. Because the set $L_{p,q}$ is regular, there is a number $n_{p,q} \geq 0$, vectors $\boldsymbol{c_{p,q,1}}, \boldsymbol{c_{p,q,2}}, \ldots, \boldsymbol{c_{p,q,n_{p,q}}} \in \mathbb{N}^k$, and finite $P_{p,q,1}, P_{p,q,2}, \ldots, P_{p,q,n_{p,q}} \subset \mathbb{N}^k \setminus \{\mathbf{0}\}$ with $\psi(L_{p,q}) = \bigcup_{i=1}^{n_{p,q}} \mathsf{L}(\boldsymbol{c_{p,q,i}}, P_{p,q,i})$. Set $n_0 = \max\left(\{\, n_{f,g} \mid f, g \in F \,\}\right)$. For $i \in \{1, 2, \ldots, n_{p,q}\}$ let $P_{p,q,i}$

be the set $\{\boldsymbol{x_{p,q,1}}, \boldsymbol{x_{p,q,2}}, \ldots, \boldsymbol{x_{p,q,|P_{p,q,i}|}}\}$ and $\kappa_{p,q,i} : \mathbb{N}^{|P_{p,q,i}|} \to \mathbb{N}^k$ be the map which is defined through

$$\left(\lambda_1, \lambda_2, \ldots, \lambda_{|P_{p,q,i}|}\right) \mapsto \sum_{j=1}^{|P_{p,q,i}|} \lambda_j \boldsymbol{x_{p,q,j}}.$$

For $f, g \in F$, $i \in \{1, 2, \ldots, n_{f,g}\}$, and $R \in \mathbb{Z}$ let

$$A_{f,g,i,R} = \left\{ \boldsymbol{\lambda} \in \mathbb{N}^{|P_{f,g,i}|} \;\middle|\; \boldsymbol{c_{f,g,i}} + \kappa_{f,g,i}(\boldsymbol{\lambda}) \in \mathsf{L}(\boldsymbol{0}, \{\boldsymbol{v}\}) + R\boldsymbol{w} \right\}.$$

Set

$$X = \left\{ (f,g,i) \in F \times F \times \mathbb{N} \;\middle|\; 1 \le i \le n_{f,g}, \forall r \in \mathbb{N} : \bigcup_{R \in \mathbb{Z}, |R| \ge r} A_{f,g,i,R} \ne \emptyset \right\}$$

and let r_0 be the maximum of

$$\{0\} \cup \{\, |R| \mid R \in \mathbb{Z}, \exists (f,g,i) \in (F \times F \times \mathbb{N}) \setminus X : (1 \le i \le n_{f,g} \wedge A_{f,g,i,R} \ne \emptyset) \,\}.$$

For $f, g \in F$ and $i \in \{1, 2, \ldots, n_{f,g}\}$ let $D_{f,g,i}$ be the set of minimal elements regarding the $\le$-relation on $\mathbb{N}^{|P_{p,q,i}|}$ of $\bigcup_{R \in \mathbb{Z}} A_{f,g,i,R}$. Due to [7] the set $D_{f,g,i}$ is finite. Let $r_{f,g,i}$ be the maximum of

$$\{0\} \cup \{\, |R| \mid R \in \mathbb{Z}, \exists \boldsymbol{\lambda} \in D_{f,g,i} : \boldsymbol{c_{f,g,i}} + \kappa_{f,g,i}(\boldsymbol{\lambda}) \in \mathsf{L}(\boldsymbol{0}, \{\boldsymbol{v}\}) + R\boldsymbol{w} \,\}.$$

For $(f,g,i) \in X$ there is an $R \in \mathbb{Z}$ with $|R| > r_{f,g,i}$ and $A_{f,g,i,R} \ne \emptyset$. Moreover, for vector $\boldsymbol{\lambda} = \left(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{|P_{f,g,i}|}\right) \in A_{f,g,i,R}$ there are $\boldsymbol{\mu} = \left(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_{|P_{f,g,i}|}\right) \in D_{f,g,i}$ and $R' \in \mathbb{Z}$ with $\boldsymbol{\mu} \le \boldsymbol{\lambda}$ and $\boldsymbol{\mu} \in A_{f,g,i,R'}$. It follows $|R'| \le r_{f,g,i} < |R|$ and that there exists a $T \in \mathbb{Z}$ with

$$\kappa_{f,g,i}(\boldsymbol{\lambda} - \boldsymbol{\mu}) = (\boldsymbol{c_{f,g,i}} + \kappa_{f,g,i}(\boldsymbol{\lambda})) - (\boldsymbol{c_{f,g,i}} + \kappa_{f,g,i}(\boldsymbol{\mu})) = T\boldsymbol{v} + (R - R')\boldsymbol{w}.$$

So, for all $(f,g,i) \in X$ there are $\boldsymbol{\lambda_{f,g,i}} \in \mathbb{N}^{|P_{f,g,i}|}$, $T_{f,g,i} \in \mathbb{Z}$, and $R_{f,g,i} \in \mathbb{Z} \setminus \{0\}$ with $\kappa_{f,g,i}(\boldsymbol{\lambda_{f,g,i}}) = T_{f,g,i} \cdot \boldsymbol{v} + R_{f,g,i} \cdot \boldsymbol{w}$. Let r_1 be the maximum of $\{1\} \cup \{\, |R_{f,g,i}| \mid (f,g,i) \in X \,\}$ and t_0 be the maximum of $\{0\} \cup \{\, |T_{f,g,i}| \mid (f,g,i) \in X \,\}$.

We will now define the words $\alpha_m \in \Sigma^*$ for $m \in \{0, 1, \ldots, |S'|\}$ recursively. Set $\alpha_0 = \lambda$. Let now $m \in \{1, 2, \ldots, |S'|\}$, assume that α_{m-1} is already defined, and let $p_m \in Q$ and $\beta_m \in \Sigma^*$ such that $s_m \alpha_{m-1} \circlearrowright^{|\alpha_{m-1}|} p_m \beta_m$. If

$$\bigcup_{f \in F} \left(\psi(L_{p_m,f}) \cap (\mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) - \psi(\alpha_{m-1}) + \psi(\beta_m))\right) = \emptyset,$$

we set $\alpha_m = \alpha_{m-1}$. Otherwise, let $f_m \in F$ with

$$\psi(L_{p_m,f_m}) \cap (\mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) - \psi(\alpha_{m-1}) + \psi(\beta_m)) \ne \emptyset$$

and let

$$\gamma_m \in L_{p_m,f_m} \cap \psi^{-1}\left((\mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) - \psi(\alpha_{m-1}) + \psi(\beta_m)) \cap \mathbb{N}^k\right).$$

It follows $s_m \alpha_{m-1} \gamma_m \circlearrowright^{|\alpha_{m-1}\gamma_m|} f_m \beta_m$. If there are no $g \in F$ and $i \in \mathbb{N}$ such that $(f_m, g, i) \in X$, we set $\alpha_m = \alpha_{m-1}\gamma_m$. Otherwise, let $g_m \in F$ and $i_m \in \mathbb{N}$ with $(f_m, g_m, i_m) \in X$. Furthermore, let $\delta_m \in L_{f_m,g_m}$ such that

$$\psi(\delta_m) = \boldsymbol{c_{f_m,g_m,i_m}} + |S'| \cdot |F| \cdot n_0 \cdot (2r_0 + 1) \cdot \frac{r_1!}{|R_{f,g,i}|} \cdot \kappa_{f_m,g_m,i_m}(\boldsymbol{\lambda_{f_m,g_m,i_m}})$$

and set $\alpha_m = \alpha_{m-1}\gamma_m\delta_m$. It follows $s_m\alpha_m \circlearrowleft^{|\alpha_m|} g_m\beta_m$. This completes the recursive definition of the words α_m, for $m \in \{0, 1, \ldots, |S'|\}$. Since $\pi_{k,\{j\}}(\boldsymbol{v}) > 0$ for all $j \in \{1, 2, \ldots, k\}$, there is a $\zeta \in \Sigma^*$ such that $\psi\left(\alpha_{|S'|}\zeta\right) \in L\left(\boldsymbol{0}, \{\boldsymbol{v}\}\right)$. Let $\eta \in \psi^{-1}(\{\boldsymbol{w}\})$ and, for all $m \in \{0, 1, \ldots, |S'|\}$, let $\alpha'_m \in \Sigma^*$ such that $\alpha_m\alpha'_m = \alpha_{|S'|}$.

Let now $m \in \{1, 2, \ldots, |S'|\}$ with

$$\bigcup_{f\in F} \left(\psi(L_{p_m,f}) \cap (\mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) - \psi(\alpha_{m-1}) + \psi(\beta_m))\right) = \emptyset$$

and $K > 0$. It holds

$$\begin{aligned}\psi\left(\alpha'_m\zeta\eta^{K\cdot r_1!}\beta_m\right) &= \psi\left(\alpha_{|S'|}\zeta\eta^{K\cdot r_1!}\right) - \psi(\alpha_{m-1}) + \psi(\beta_m)\\ &\in \mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) - \psi(\alpha_{m-1}) + \psi(\beta_m).\end{aligned}$$

We have

$$s_m\alpha_{|S'|}\zeta\eta^{K\cdot r_1!} = s_m\alpha_{m-1}\alpha'_m\zeta\eta^{K\cdot r_1!} \circlearrowleft^{|\alpha_{m-1}|} p_m\alpha'_m\zeta\eta^{K\cdot r_1!}\beta_m$$

and $\psi(\alpha'_m\zeta\eta^{K\cdot r_1!}\beta_m) \notin \psi(L_{p_m,f})$ for all $f \in F$. With an analogous argument as in the proof of Lemma 2 in [2] it follows $\alpha_{|S'|}\zeta\eta^{K\cdot r_1!} \notin L_R((Q, \Sigma, R, \{s_m\}, F))$.

Now, let $m \in \{1, 2, \ldots, |S'|\}$ such that

$$\bigcup_{f\in F} \left(\psi(L_{p_m,f}) \cap (\mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) - \psi(\alpha_{m-1}) + \psi(\beta_m))\right) \neq \emptyset$$

and there are no $g \in F$ and $i \in \mathbb{N}$ with $(f_m, g, i) \in X$. For all $g \in F$, $i \in \{1, 2, \ldots, n_{f_m,g}\}$, and $R \in \mathbb{Z}$ with $|R| > r_0$ we have $A_{f_m,g,i,R} = \emptyset$. For all $K > 0$ it holds

$$s_m\alpha_{|S'|}\zeta\eta^{K\cdot r_1!} = s_m\alpha_{m-1}\gamma_m\alpha'_m\zeta\eta^{K\cdot r_1!} \circlearrowleft^{|\alpha_{m-1}\gamma_m|} f_m\alpha'_m\zeta\eta^{K\cdot r_1!}\beta_m,$$

and

$$\begin{aligned}\psi\left(\alpha'_m\zeta\eta^{K\cdot r_1!}\beta_m\right) &= \psi\left(\alpha_{|S'|}\zeta\eta^{K\cdot r_1!}\right) - (\psi(\gamma_m) + \psi(\alpha_{m-1}) - \psi(\beta_m))\\ &= \psi\left(\alpha_{|S'|}\zeta\right) - (\psi(\gamma_m) + \psi(\alpha_{m-1}) - \psi(\beta_m)) + K \cdot r_1! \cdot \boldsymbol{w},\end{aligned}$$

where $\psi\left(\alpha_{|S'|}\zeta\right) \in \mathsf{L}(\boldsymbol{0}, \{\boldsymbol{v}\})$ and $\psi(\gamma_m) + \psi(\alpha_{m-1}) - \psi(\beta_m) \in \mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\})$. If the element $\alpha_{|S'|}\zeta\eta^{K\cdot r_1!} \in L_R((Q, \Sigma, R, \{s_m\}, F))$ for a $K > 0$, we have

$$\psi\left(\alpha'_m\zeta\eta^{K\cdot r_1!}\beta_m\right) \in \bigcup_{g\in F} \bigcup_{i=1}^{n_{f_m,g}} \mathsf{L}(\boldsymbol{c_{f_m,g,i}}, P_{f_m,g,i}).$$

It follows

$$\left|\left\{ \alpha_{|S'|}\zeta\eta^{K\cdot r_1!} \;\middle|\; K > 0 \right\} \cap L_R((Q, \Sigma, R, \{s_m\}, F))\right| \leq |F| \cdot n_0 \cdot (2r_0 + 1).$$

Because of $\psi\left(\alpha_{|S'|}\zeta\eta^{K\cdot r_1!}\right) \in \mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) \subseteq S$ for all $K > 0$, there are a $K > 0$ with

$$K \leq \left(|S'| - 1\right) \cdot |F| \cdot n_0 \cdot (2r_0 + 1) + 1 \leq |S'| \cdot |F| \cdot n_0 \cdot (2r_0 + 1)$$

and an $m \in \{1, 2, \ldots, |S'|\}$ such that

$$\bigcup_{f\in F} \left(\psi(L_{p_m,f}) \cap (\mathsf{L}(\boldsymbol{w}, \{\boldsymbol{v}, \boldsymbol{w}\}) - \psi(\alpha_{m-1}) + \psi(\beta_m))\right) \neq \emptyset,$$

$\alpha_m \neq \alpha_{m-1}\gamma_m$, and $\alpha_{|S'|}\zeta\eta^{K\cdot r_1!} \in L_R((Q,\Sigma,R,\{s_m\},F))$. We have

$$s_m\alpha_{|S'|}\zeta\eta^{K\cdot r_1!} \circlearrowright^{|\alpha_m|} g_m\alpha'_m\zeta\eta^{K\cdot r_1!}\beta_m.$$

Let $\delta'_m \in L_{f_m,g_m}$ such that $\psi(\delta'_m)$ equals

$$\boldsymbol{c_{f_m,g_m,i_m}} + \left(|S'|\cdot|F|\cdot n_0\cdot(2r_0+1) - \frac{R_{f,g,i}}{|R_{f,g,i}|}\cdot K\right)\cdot\frac{r_1!}{|R_{f,g,i}|}\cdot\kappa_{f_m,g_m,i_m}(\boldsymbol{\lambda_{f_m,g_m,i_m}}).$$

We get

$$s_m\alpha_{m-1}\gamma_m\delta'_m\alpha'_m\zeta\eta^{K\cdot r_1!} \circlearrowright^{|\alpha_{m-1}\gamma_m\delta'_m|} g_m\alpha'_m\zeta\eta^{K\cdot r_1!}\beta_m,$$

which implies

$$\alpha_{m-1}\gamma_m\delta'_m\alpha'_m\zeta\eta^{K\cdot r_1!} \in L_R((Q,\Sigma,R,\{s_m\},F)) \subseteq \psi^{-1}(S).$$

It holds

$$\begin{aligned}
\psi\left(\alpha_{m-1}\gamma_m\delta'_m\alpha'_m\zeta\eta^{K\cdot r_1!}\right) &- \psi\left(\alpha_{|S'|}\zeta\eta^{K\cdot r_1!}\right)\\
&= \psi(\delta'_m) - \psi(\delta_m)\\
&= -\frac{R_{f,g,i}}{|R_{f,g,i}|}\cdot K\cdot\frac{r_1!}{|R_{f,g,i}|}\cdot\kappa_{f_m,g_m,i_m}(\boldsymbol{\lambda_{f_m,g_m,i_m}})\\
&= \frac{-K\cdot r_1!}{R_{f,g,i}}\cdot(T_{f,g,i}\cdot\boldsymbol{v} + R_{f,g,i}\cdot\boldsymbol{w})\\
&= \frac{-K\cdot r_1!\cdot T_{f,g,i}}{R_{f,g,i}}\cdot\boldsymbol{v} - K\cdot r_1!\cdot\boldsymbol{w}.
\end{aligned}$$

Because of $\psi\left(\alpha_{|S'|}\zeta\eta^{K\cdot r_1!}\right) \in L(K\cdot r_1!\cdot\boldsymbol{w},\{\boldsymbol{v}\})$, we get

$$\psi\left(\alpha_{m-1}\gamma_m\delta'_m\alpha'_m\zeta\eta^{K\cdot r_1!}\right) \in \mathsf{L}(\boldsymbol{0},\{\boldsymbol{v}\}),$$

which gives us $\psi\left(\alpha_{m-1}\gamma_m\delta'_m\alpha'_m\zeta\eta^{K\cdot r_1!}\right) \notin S$, a contradiction. This proves the lemma. □

Because the Parikh-image of $\{\, w \in \{a,b\}^* \mid |w|_a \neq |w|_b \,\}$ is a 0-anti-lattice, this language is not in **MROWJ**. Thus, we have **pMROWJ** $\subset$ **JFA** and that the family **pMROWJ** is incomparable to **pDCF** and to **pCF**.

To extend Theorem 20 to arbitrary anti-lattices, we need the following:

Lemma 21. *Let Σ be an alphabet, $a \in \Sigma$, and $L \subseteq \Sigma^*$ be a language from* **pMROWJ**. *Then, the language $\{\, vaw \mid v,w \in \Sigma^*, vw \in L \,\}$ is also contained in the language family* **pMROWJ**.

Proof. Let $A = (Q,\Sigma,R,S,F)$ be an MROWJFA with $L_R(A) = L$. Consider the MROWJFA $A' = (Q\cup S',\Sigma,R',S',F)$, where $S' = \{\, s' \mid s \in S \,\}$ and for all $(q,b) \in Q\times\Sigma$ the value $R'(q,b)$ is defined if and only if $R(q,b)$ is defined. In this case we have $R'(q,b) = R(q,b)$. For $s \in S$ we get $R'(s',a) = s$ and for all $b \in \Sigma\setminus\{a\}$ the value $R'(s',b)$ is undefined. Obviously, we have $|w|_a > 0$ for every $w \in L_R(A')$. For $v,w \in \Sigma^*$ with $|v|_a = 0$ and $s \in S$ it holds $s'vaw \circlearrowright^{|v|}_{A'} s'awv \circlearrowright_{A'} swv$. This gives us

$$vaw \in L_R(A') \Leftrightarrow wv \in L_R(A) \Leftrightarrow vw \in L_R(A),$$

which implies

$$L_R(A') = \{\, vaw \mid v,w \in \Sigma^*, |v|_a = 0, vw \in L \,\} = \{\, vaw \mid v,w \in \Sigma^*, vw \in L \,\}.$$

That proves the lemma. □

From Theorem 20 and Lemmas 15 and 21 we get:

Corollary 22. *The family* **pMROWJ** *does not contain any language whose Parikh-image is an anti-lattice.*

Proposition 16 and 22 give us:

Corollary 23. *The Parikh-image of each language in* **pMROWJ** *is a quasi lattice.*

To get more detailed results about **pMROWJ**, we define the language operation of *disjoint quotient* of a language $L \subseteq \Sigma^*$ with a word $w \in \Sigma^*$ as follows:

$$\begin{aligned} L/^d w &= \{\, v \in \Sigma^* \mid vw \in L, \forall a \in \Sigma : (|v|_a = 0 \vee |w|_a = 0) \,\} \\ &= (L/w) \cap \{\, a \in \Sigma \mid |w|_a = 0 \,\}^*. \end{aligned}$$

From Theorem 4 we get that the family **pROWJ** is closed under the operations of quotient with a word and disjoint quotient with a word. Let Σ be an alphabet, $\Pi \subseteq \Sigma$, and $L \subseteq \Sigma^*$ be in **MROWJ**. Then, it is easy to see that $L \cap \Pi^*$ is also in **MROWJ**. Thus, we get that if **pMROWJ** is closed under the operation quotient with a word, then **pMROWJ** is also closed under disjoint quotient with a word.

Theorem 4 gave a characterization of the language class **pROWJ** in terms of the Myhill-Nerode relation. The next Corollary is a result in the same spirit for the permutation closed language family **pMROWJ**. Theorems 4 and 18 and Corollary 23 give us a characterization of all languages L for which each disjoint quotient of L with a word is contained in **pMROWJ**:

Corollary 24. *For an alphabet Σ and a permutation closed language $L \subseteq \Sigma^*$ the following conditions are equivalent:*

1. *For all $w \in \Sigma^*$ the language $L/^d w$ is in* **pMROWJ**.
2. *There is an $n \geq 0$ and $L_1, L_2, \ldots, L_n \subseteq \Sigma^*$ with $L_1, L_2, \ldots, L_n \in$* **pROWJ** *and $L = \bigcup_{i=1}^{n} L_i$.*
3. *There is an $n \geq 0$ and permutation closed languages $L_1, L_2, \ldots, L_n \subseteq \Sigma^*$ such that language $L = \bigcup_{i=1}^{n} L_i$ and for all $i \in \{1, 2, \ldots, n\}$ the language L_i has only a finite number of positive Myhill-Nerode equivalence classes.*
4. *There is an $m \geq 0$ and linear sets $L_1, L_2, \ldots, L_m \subseteq \mathbb{N}^{|\Sigma|}$ such that $\psi(L) = \bigcup_{j=1}^{m} L_j$ and for each $j \in \{1, 2, \ldots, m\}$ we have $\left|L_j / \equiv_{L_j}\right| = 1$.*
5. *The set $\pi_{|\Sigma|,\{1,2,\ldots,|\Sigma|\}\setminus T}\left(\{\, \boldsymbol{z} \in \psi(L) \mid \pi_{|\Sigma|,T}(\boldsymbol{z}) = \boldsymbol{x} \,\}\right)$ is a quasi lattice, for all subsets $T \subseteq \{1, 2, \ldots, |\Sigma|\}$ and vectors $\boldsymbol{x} \in \mathbb{N}^{|T|}$.*

For ternary alphabets we can weaken the first condition of the previous corollary, by the fact that the family **JFA** is closed under the operation of disjoint quotient, and strengthen its fourth condition by Corollary 19:

Corollary 25. *For an alphabet Σ with $|\Sigma| = 3$ and a permutation closed language $L \subseteq \Sigma^*$ the following two conditions are equivalent:*

1. *For all unary $w \in \Sigma^*$ the language $L/^d w$ is in* **pMROWJ**.
2. *There is a number $m \geq 0$, vectors $\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_m} \in \mathbb{N}^3$, and linearly independent subsets $P_1, P_2, \ldots, P_m \subseteq \mathbb{N}^3$ such that $\psi(L) = \bigcup_{j=1}^{m} \mathsf{L}(\boldsymbol{c_j}, P_j)$ and $\left|\mathsf{L}(\boldsymbol{0}, P_j) / \equiv_{\mathsf{L}(\boldsymbol{0}, P_j)}\right| = 1$, for each $j \in \{1, 2, \ldots, m\}$.*

From Theorem 4 and Corollary 24 we get that the condition that each language from **pMROWJ** is a finite union of languages from **pROWJ** is equivalent to the condition that the family **pMROWJ** is closed under the operation of quotient with a word and to the condition that the family **pMROWJ** is closed under the operation of disjoint quotient with a word.

Consider an alphabet Σ and a language $L \subseteq \Sigma^*$. If for all $w \in \Sigma^*$ the language $L/^d w$ is in **pMROWJ**, then the language L is contained in the complexity class **DTIME**(n), as the next result shows:

Lemma 26. *Let Σ be an alphabet, $n > 0$, and $L_1, L_2, \ldots, L_n \subseteq \Sigma^*$ be in* **pROWJ**. *Then, there is a one-way $(n \cdot |\Sigma|)$-head DFA with endmarker accepting $\bigcup_{j=1}^{n} L_i$.*

Proof. Let $\Sigma = \{a_1, a_2, \ldots, a_{|\Sigma|}\}$ and for $j \in \{1, 2, \ldots, n\}$ let A_j be a ROWJFA accepting L_j. We will informally describe how a one-way $|\Sigma|$-head DFA with endmarker accepting L_j works. At the beginning for each $i \in \{1, 2, \ldots, |\Sigma|\}$ the i-th head moves to the first occurrence of a_i. In each step of the computation in which not all heads have reached the endmarker yet, the automaton determines all $i \in \{1, 2, \ldots, |\Sigma|\}$ for which A_j can read a_i in the current state and the i-th head has not reached the endmarker yet. If there is no such i, the input is rejected. Otherwise, let i_0 be the lowest such i. The automaton reads a_{i_0}, changes the state according to A_j, and moves the i_0-th head to the next occurrence of a_{i_0}. When all heads have reached the endmarker, the input is accepted if and only if the automaton is in an accepting state. Because L_j is closed under permutation, the accepted language is exactly L_j. Simulating the n one-way $|\Sigma|$-head DFAs with endmarker that accept the languages L_j for $j \in \{1, 2, \ldots, n\}$ simultaneously, we get a one-way $(n \cdot |\Sigma|)$-head DFA with endmarker accepting $\bigcup_{j=1}^{n} L_i$. $\square$

4.2 Results for Binary Alphabets

Now, we will investigate **MROWJ** and related language families for binary alphabets. It turns out that for some problems we get different or stronger results than for arbitrary alphabets. From the next theorem it follows that for binary alphabets **pCF** = **JFA**, whereas for arbitrary alphabets it holds **pCF** $\subset$ **JFA**.

Theorem 27. *Each permutation closed semi-linear language over a binary alphabet is accepted by a counter automaton.*

Proof. Let $\psi : \{a, b\}^* \to \mathbb{N}^2$ be the Parikh-mapping. Since the family of languages accepted by counter automata is closed under the operation of union and each semi-linear set can be written as a finite union of linear sets with linearly independent periods by Theorem 2, it suffices to show that for $c, d, e, f, g, h \in \mathbb{N}$ the language

$$\psi^{-1}\left(\left\{ \begin{pmatrix} c \\ d \end{pmatrix} + \lambda \begin{pmatrix} e \\ f \end{pmatrix} + \mu \begin{pmatrix} g \\ h \end{pmatrix} \;\middle|\; \lambda, \mu \in \mathbb{N} \right\}\right)$$

is accepted by a counter automaton. To do so, define the maps $M, N : \{p, q, r\} \to \mathbb{N}$ through $M(p) = c$, $M(q) = e$, $M(r) = g$, $N(p) = d$, $N(q) = f$, and $N(r) = h$. Let

$$Q = \{t\} \cup \bigcup_{s \in \{p,q,r\}} \{\, s_{m,n} \mid m \in \{0, 1, \ldots, M(s)\}, n \in \{0, 1, \ldots, N(s)\} \,\}$$

and $\Gamma = \{\perp, a, b\}$ and consider the pushdown automaton A given by

$$A = (Q, \{a, b\}, \Gamma, \delta, p_{0,0}, \perp, \{t\}).$$

The map $\delta : Q \times \{a, b, \lambda\} \times \Gamma \to 2^{Q\times\left(\bigcup_{i=0}^{2}\Gamma^i\right)}$ is given by

$$\delta = \{\, ((t, y, x), \emptyset) \mid y \in \{a, b, \lambda\}, x \in \Gamma \,\} \cup \bigcup_{s\in\{p,q,r\}} \delta_s,$$

where, for $s \in \{p, q, r\}$, the map δ_s, that maps from

$$\{\, s_{m,n} \mid m \in \{0, 1, \ldots, M(s)\}, n \in \{0, 1, \ldots, N(s)\} \,\} \times \{a, b, \lambda\} \times \Gamma$$

to $2^{Q\times\left(\bigcup_{i=0}^{2}\Gamma^i\right)}$, is defined as

$$\begin{aligned}
\delta_s = &\{\, ((s_{m,n}, \lambda, a), \{(s_{m+1,n}, \lambda)\}) \mid m \in \{0, 1, \ldots, M(s) - 1\}, n \in \{0, 1, \ldots, N(s)\} \,\} \\
&\cup \left\{\, \left((s_{M(s),n}, \lambda, a), \emptyset\right) \;\middle|\; n \in \{0, 1, \ldots, N(s) - 1\} \,\right\} \\
&\cup \{\, ((s_{m,n}, a, x), \{(s_{m+1,n}, x)\}) \mid x \in \Gamma, m \in \{0, 1, \ldots, M(s) - 1\}, n \in \{0, 1, \ldots, N(s)\} \,\} \\
&\cup \left\{\, \left((s_{M(s),n}, a, x), \{(s_{M(s),n}, xa)\}\right) \;\middle|\; x \in \{a, \perp\}, n \in \{0, 1, \ldots, N(s)\} \,\right\} \\
&\cup \left\{\, \left((s_{M(s),n}, a, b), \emptyset\right) \;\middle|\; n \in \{0, 1, \ldots, N(s)\} \,\right\} \\
&\cup \{\, ((s_{m,n}, \lambda, b), \{(s_{m,n+1}, \lambda)\}) \mid m \in \{0, 1, \ldots, M(s)\}, n \in \{0, 1, \ldots, N(s) - 1\} \,\} \\
&\cup \left\{\, \left((s_{m,N(s)}, \lambda, b), \emptyset\right) \;\middle|\; m \in \{0, 1, \ldots, M(s) - 1\} \,\right\} \\
&\cup \{\, ((s_{m,n}, b, x), \{(s_{m,n+1}, x)\}) \mid x \in \Gamma, m \in \{0, 1, \ldots, M(s)\}, n \in \{0, 1, \ldots, N(s) - 1\} \,\} \\
&\cup \left\{\, \left((s_{m,N(s)}, b, x), \{(s_{m,N(s)}, xb)\}\right) \;\middle|\; x \in \{b, \perp\}, m \in \{0, 1, \ldots, M(s)\} \,\right\} \\
&\cup \left\{\, \left((s_{m,N(s)}, b, a), \emptyset\right) \;\middle|\; m \in \{0, 1, \ldots, M(s)\} \,\right\} \\
&\cup \left\{\, \left((s_{M(s),N(s)}, \lambda, x), \{(q_{0,0}, x), (r_{0,0}, x)\}\right) \;\middle|\; x \in \{a, b\} \,\right\} \\
&\cup \left\{\left((s_{M(s),N(s)}, \lambda, \perp), \{(q_{0,0}, \perp), (r_{0,0}, \perp), (t, \perp)\}\right)\right\} \\
&\cup \{\, ((s_{m,n}, \lambda, \perp), \emptyset) \mid (m, n) \in (\{0, 1, \ldots, M(s)\} \times \{0, 1, \ldots, N(s)\}) \setminus \{(M(s), N(s))\} \,\}\,.
\end{aligned}$$

For $s \in \{p, q, r\}$, $w, w' \in \{a, b\}^*$, $v, v' \in \Gamma^*$, $m, m' \in \{0, 1, \ldots, M(s)\}$, and $n, n' \in \{0, 1, \ldots, N(s)\}$ with $m' + n' \geq m + n$ and $(s_{m,n}, w, v) \vdash_A (s_{m',n'}, w', v')$, we write $(s_{m,n}, w, v) \nearrow (s_{m',n'}, w', v')$. Let $s \in \{p, q, r\}$. For $k \in \{0, 1, \ldots, M(s)\}$ it holds $(s_{0,0}, \lambda, \perp a^k) \nearrow^k (s_{k,0}, \lambda, \perp)$, while for $k \in \{0, 1, \ldots, N(s)\}$ we have $(s_{0,0}, \lambda, \perp b^k) \nearrow^k (s_{0,k}, \lambda, \perp)$. So, for $w \in \{a, b\}^*$ and $k, \ell \geq 0$ we get

$$\begin{aligned}
(s_{0,0}, w, \perp a^k) \nearrow^* (s_{M(s),N(s)}, \lambda, \perp a^\ell) \quad &\Leftrightarrow \quad \psi(w) = \begin{pmatrix} M(s) + \ell - k \\ N(s) \end{pmatrix}, \\
(s_{0,0}, w, \perp a^k) \nearrow^* (s_{M(s),N(s)}, \lambda, \perp b^\ell) \quad &\Leftrightarrow \quad \psi(w) = \begin{pmatrix} M(s) - k \\ N(s) + \ell \end{pmatrix}, \\
(s_{0,0}, w, \perp b^k) \nearrow^* (s_{M(s),N(s)}, \lambda, \perp a^\ell) \quad &\Leftrightarrow \quad \psi(w) = \begin{pmatrix} M(s) + \ell \\ N(s) - k \end{pmatrix}, \\
(s_{0,0}, w, \perp b^k) \nearrow^* (s_{M(s),N(s)}, \lambda, \perp b^\ell) \quad &\Leftrightarrow \quad \psi(w) = \begin{pmatrix} M(s) \\ N(s) + \ell - k \end{pmatrix}.
\end{aligned}$$

For all $w \in \{a, b\}^*$, $x \in \{a, b\}$, and $k \geq 0$ with

$$\psi(x^k w) \in \left\{ \begin{pmatrix} M(s) + \ell_1 \\ N(s) + \ell_2 \end{pmatrix} \;\middle|\; \ell_1, \ell_2 \geq 0 \right\}$$

there is a prefix v of w such that

$$\psi(x^k v) \in \left\{ \begin{pmatrix} M(s) + \ell_1 \\ N(s) + \ell_2 \end{pmatrix} \;\middle|\; \ell_1, \ell_2 \geq 0 \text{ with } \ell_1 \ell_2 = 0 \right\}.$$

This gives us

$$L(A) = \psi^{-1}\left(\left\{ \begin{pmatrix} c \\ d \end{pmatrix} + \lambda \begin{pmatrix} e \\ f \end{pmatrix} + \mu \begin{pmatrix} g \\ h \end{pmatrix} \;\middle|\; \lambda, \mu \in \mathbb{N} \right\}\right).$$

In all situations there are no a's or no b's on the stack, so A can be simulated by a counter automaton, which proves the theorem. $\square$

For binary alphabets we have **pROWJ** $\subset$ **pDCF**, while for arbitrary alphabets **pROWJ** is incomparable to **pDCF** and to **pCF**:

Proposition 28. *Each language over a binary alphabet in* **pROWJ** *is accepted by a realtime deterministic counter automaton.*

Proof. Let Σ be an alphabet with $|\Sigma| = 2$ and $L \subseteq \Sigma^*$ be in **pROWJ**. Let $A = (Q, \Sigma, R, \{s\}, F)$ be a ROWJFA with $L_R(A) = L$. Extend the partial function $R : Q \times \Sigma \to Q$ to $Q \times \Sigma^*$ in the natural way. Let the map $c : Q \times \Sigma \to \{0, 1\}$ be defined as follows. For $q \in Q$ and $a, b \in \Sigma$ with $a \neq b$ we have $c(q, a) = 1$ if and only if there is an $n \geq 0$ such that $R(q, b^n a)$ is defined. For $q \in Q$ and $a, b \in \Sigma$ with $a \neq b$ and $c(q, a) = 1$ let $n(q, a)$ be the lowest $n \geq 0$ for which $R(q, b^n a)$ is defined.

Consider the deterministic pushdown automaton

$$B = (Q \cup F', \Sigma, \Sigma \cup \{\perp\}, \delta, s, \perp, F')$$

where $F' = \{\, f' \mid f \in F \,\}$ and the partial map

$$\delta : (Q \cup F') \times (\Sigma \cup \{\lambda\}) \times (\Sigma \cup \{\perp\}) \to (Q \cup F') \times (\Sigma \cup \{\perp\})^*$$

is defined as follows:

- For $q \in Q \setminus F$ and $a, b \in \Sigma$ with $a \neq b$ let $\delta(q, a, \perp) = \left(R\left(q, b^{n(q,a)}a\right), \perp b^{n(q,a)}\right)$ if $c(q, a) = 1$.
- For $f \in F$ and $a, b \in \Sigma$ with $a \neq b$ let $\delta(f', a, \perp) = \left(R\left(f, b^{n(f,a)}a\right), \perp b^{n(f,a)}\right)$ if $c(f, a) = 1$.
- For $f \in F$ we have $\delta(f, \lambda, \perp) = (f', \perp)$.
- For $q \in Q$ and $a \in \Sigma$ let $\delta(q, a, a) = (q, \lambda)$.
- For $q \in Q$ and $a, b \in \Sigma$ with $a \neq b$ let $\delta(q, a, b) = \left(R\left(q, b^{n(q,a)}a\right), b^{n(q,a)+1}\right)$ if $c(q, a) = 1$.

With induction over $|v|$ one can see that for all $v \in \Sigma^*$, $q \in Q$, $a \in \Sigma$, and $n \geq 0$ it holds that if from the configuration $(s, v, \perp)$ the configuration (q, λ, a^n) can be reached with B, then it holds that there is a permutation x of va^n with $R(s, x) = q$ and that for each $m \geq 0$ with $m < n$ there is a $y \in \Sigma^*$ such that yb is a permutation of va^m, the value $R(s, y)$ is defined, and $R(s, yb)$ is undefined. Because L is closed under permutation, we get $L(B) = L$. It should be clear that B can be simulated by a realtime deterministic pushdown automaton by not writing the symbol that would be located directly above $\perp$ on the stack, but remembering this symbol in the finite control of the automaton. Except for the bottom symbol, in all situations there are no two different symbols on the stack, so B can even be simulated by a realtime deterministic counter automaton. $\square$

For the family **pMROWJ** we get the following results. Notice that for arbitrary alphabets **pMROWJ** and **pCF** are incomparable.

Corollary 29. *For binary alphabets it holds that* **pROWJ** $\subset$ **pMROWJ**, *that* **pMROWJ** *is incomparable to* **pDCF**, *and that* **pMROWJ** $\subset$ **JFA** $=$ **pCF**.

Proof. The language $\{ w \in \{a,b\}^* \mid |w|_a \neq |w|_b \}$ is contained in the language class **pDCF**, but not in **pMROWJ**, because its Parikh-image is an anti-lattice. The language

$$L = \{ w \in \{a,b\}^* \mid |w|_b = |w|_a \vee |w|_b = 2 \cdot |w|_a \}$$

is known to be not in **pDCF**. By Theorem 4 the language L is in **pMROWJ** as the union of two languages from **pROWJ**. For $n, m \geq 0$ we have $a^n b^n b^m \in L$ if and only if $m \in \{0, n\}$. So, for $n, m \geq 0$ with $n \neq m$ it holds $[a^n b^n]_L \neq [a^m b^m]_L$, which gives us that L is not in **pROWJ**. We have **pMROWJ** $\subset$ **JFA** because each language in the first class is semi-linear. $\square$

If the languages do not need to be closed under permutation, we get for binary alphabets the same inclusion relations between **ROWJ**, **MROWJ**, and **DCF** as for arbitrary alphabets:

Lemma 30. *For binary alphabets* **ROWJ** $\subset$ **MROWJ**. *The families* **ROWJ** *and* **MROWJ** *are both incomparable to* **DCF** *over binary alphabets.*

Proof. Because of Corollary 29 it suffices to show that there is a language over a binary alphabet in **ROWJ** which is not deterministic context free. Consider the ROWJFA $A = (\{q_0, q_1, \ldots, q_8\}, \{a, b\}, R, \{q_0\}, \{q_4, q_7\})$ with

$$R = \{q_0 a \to q_1, q_1 b \to q_2, q_2 a \to q_3, q_3 a \to q_4, q_4 a \to q_5, q_5 b \to q_4, \\ q_3 b \to q_6, q_6 b \to q_7, q_7 a \to q_8, q_8 b \to q_6\}.$$

For $n, m > 0$ we have

$$q_0 a^n b^m aa \circlearrowright^{n+m} q_2 aaa^{n-1} b^{m-1} \circlearrowright^2 q_4 a^{n-1} b^{m-1}.$$

For $n > 0$ and $m > 1$ it holds

$$q_0 a^n b^m ab \circlearrowright^{n+m} q_2 aba^{n-1} b^{m-1} \circlearrowright^2 q_6 a^{n-1} b^{m-1} \circlearrowright^n q_7 b^{m-2} a^{n-1}.$$

So, we get

$$L := L_R(A) \cap \left(a^+ b^+ a\{a,b\}\right) = \{ a^n b^n aa \mid n > 0 \} \cup \left\{ a^n b^{2n} ab \;\middle|\; n > 0 \right\}.$$

It follows $L/\{aa, ab\} = \{ a^n b^m \mid n > 0, m \in \{n, 2n\} \}$, which is known to be not deterministic context free. Since **DCF** is closed under intersection with regular languages and under right quotient with regular languages, the language $L_R(A)$ is also not deterministic context free. That proves the lemma. $\square$

Theorems 4, Proposition 16, and Corollaries 19 and 23 imply a characterization of the languages in **pMROWJ** over a binary alphabet, which is stronger than the statement for arbitrary alphabets in Corollary 24, because we do not need to consider disjoint quotients of a language with a word here.

Corollary 31. *Let Σ be an alphabet with $|\Sigma| = 2$ and $L \subseteq \Sigma^*$ be a permutation closed language. Then, the following conditions are equivalent:*

1. *Language L is in* **pMROWJ**.
2. *There is an $n \geq 0$ and $L_1, L_2, \ldots, L_n \subseteq \Sigma^*$ with $L_1, L_2, \ldots, L_n \in$* **pROWJ** *and $L = \bigcup_{i=1}^n L_i$.*
3. *There is an $n \geq 0$ and permutation closed languages $L_1, L_2, \ldots, L_n \subseteq \Sigma^*$ such that language $L = \bigcup_{i=1}^n L_i$ and for all $i \in \{1, 2, \ldots, n\}$ the language L_i has only a finite number of positive Myhill-Nerode equivalence classes.*

4. *There is a number $m \geq 0$, vectors $\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_m} \in \mathbb{N}^2$, and linearly independent subsets $P_1, P_2, \ldots, P_m \subseteq \mathbb{N}^2$ such that $\psi(L) = \bigcup_{j=1}^{m} \mathsf{L}(\boldsymbol{c_j}, P_j)$ and $\left|\mathsf{L}(\boldsymbol{0}, P_j)/\equiv_{\mathsf{L}(\boldsymbol{0},P_j)}\right| = 1$, for each $j \in \{1, 2, \ldots, m\}$.*
5. *The Parikh-image of L is a semi-linear set and a quasi lattice.*
6. *The Parikh-image of L is a semi-linear set and not an anti-lattice.*

From Corollary 31 it follows that each language from **pMROWJ** over a binary alphabet is a finite union of permutation closed languages accepted by a realtime deterministic counter automaton.

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