Events and languages on unary quantum automata

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Summary

- Deterministic and quantum automata (Measure-Once and Measure-Many)
- Unary regular languages
- Recognizing unary regular languages with MM-qfa's
- Periodicity decision problems on events induced by qfa's
- Transient and ergodic components of the nonhalting space
- Conclusion and open problems

Deterministic finite automata (dfa)

initial state

input alphabet

set of states

$\pi_0 \in \{0,1\}^{|Q|}$

$$DA = \langle \Sigma,\ Q,\ \pi_0,\ \{B(\sigma)\}_{\sigma\in\Sigma},\ \eta \rangle$$

transition matrices

$B(\sigma) \in \{0,1\}^{|Q|\times|Q|}$

characteristic vector of the final states

$\eta \in \{0,1\}^{|Q|}$

Language recognized by DA:

$$L_{DA} = \{w \in \Sigma^* | \pi_0 B(w)\eta = 1\}$$

where, for $w = \sigma_1 \cdots \sigma_n$, it holds $B(w) = B(\sigma_1)\cdot \ldots \cdot B(\sigma_n)$

Quantum finite automata (qfa)

- $Q = \{q_1, \ldots, q_k\}$ is the set of pure states,
- $\pi = \alpha_1 \bar{q}_1 + \ldots + \alpha_k \bar{q}_k$, such that $\|\pi\| = 1$, is the superposition of the pure states,
- $\alpha_i \in \mathbf{C}$ is the amplitude of q_i in π,
- $|\alpha_i|^2$ is the probability of observing q_i in π,
- $U(\sigma) \in \mathbf{C}^{|Q| \times |Q|}$ is the transition unitary matrix $(\pi U(\sigma) = \pi')$,
- Matrix representation of a qfa:

$$\langle \Sigma, Q, \pi_0, \{U(\sigma)\}_{\sigma \in \Sigma}, O \rangle$$

The Measure-Once model

- Matrix representation of a MO-qfa:

$$QA_{MO} = \langle \Sigma,\, Q,\, \{U(\sigma)\}_{\sigma\in\Sigma},\, O=\eta \rangle$$

- The event induced by the automaton QA_{MO} on $w=\sigma_1\cdots\sigma_n$ is

$$p_{QA}(w) = \|\pi_0 U(w)\mathrm{diag}(\eta)\|^2$$

and represents the probability of accepting the word w.

The Measure-Many model

non-halting states

- Matrix representation of a MM-qfa: halting states

$$QA_{MM} = \langle \Sigma,\, Q,\, \{U(\sigma)\}_{\sigma \in (\Sigma \cup \{\#\})},\, O = \{P_a, P_r, P_g\} \rangle$$

- The event induced by the automaton QA_{MM} on $w = \sigma_1 \cdots \sigma_n$ is

$$p_{QA}(w) \quad = \quad \sum_{i=1}^{n} \|\pi_0 \prod_{j=1}^{i-1} [U(\sigma_j) P_g] U(\sigma_i) P_a\|^2 + $$

$$+\|\pi_0 \prod_{j=1}^{n} [U(\sigma_j) P_g] U(\sharp) P_a\|^2$$

and represents the probability of accepting the word w.

Languages recognized

- Language recognized by QA with cut point $0 \leq \lambda \leq 1$:

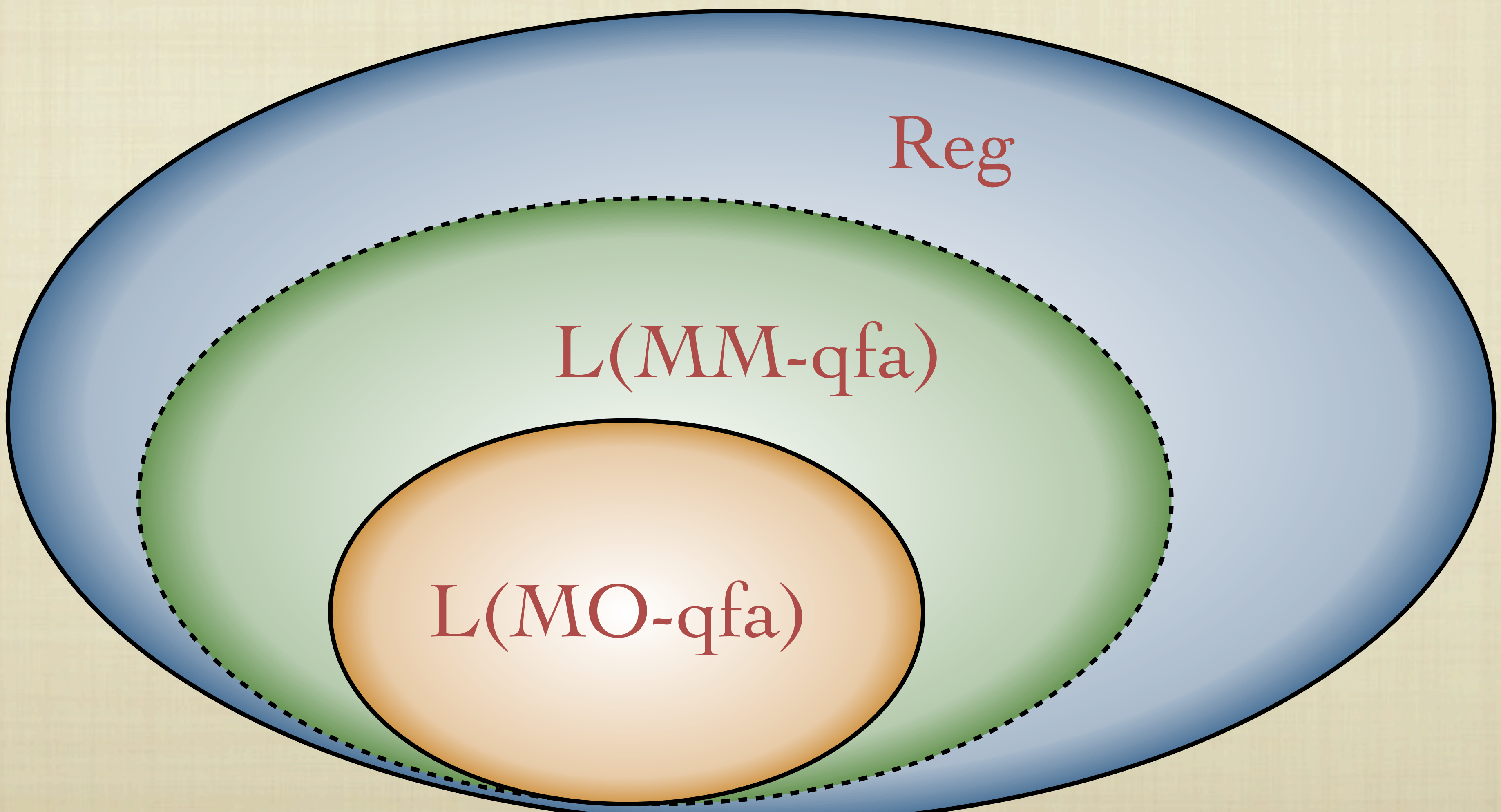
$$L_{QA} = \{w \in \Sigma^* \,|\, p_{QA}(w) > \lambda\}$$

- For $0 \leq \varepsilon \leq ½$, QA recognizes L_{QA} with cut point λ isolated by ε, if it holds

$$p_{QA}(w) \begin{cases} \geq \lambda + \varepsilon & \text{se } w \in L_{QA} \\ \leq \lambda - \varepsilon & \text{se } w \notin L_{QA} \end{cases}$$

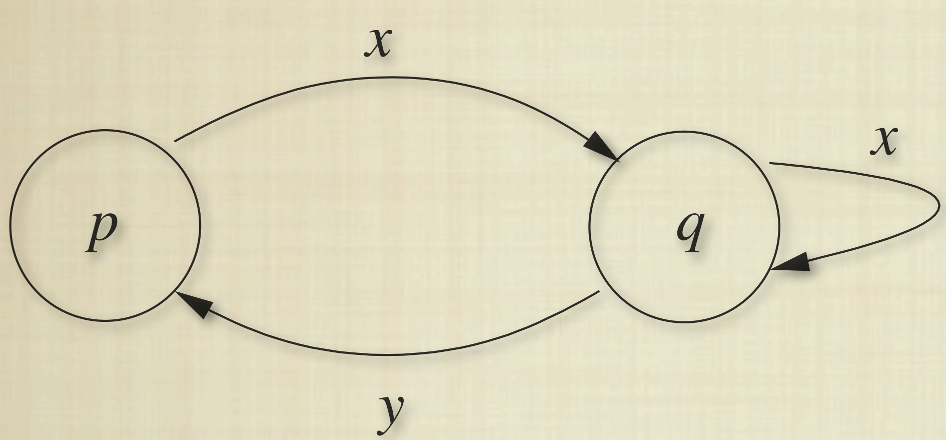
Languages recognized

- dfa: regular languages
- MO-qfa: reversible regular languages (transition=permutation)
- MM-qfa: ?

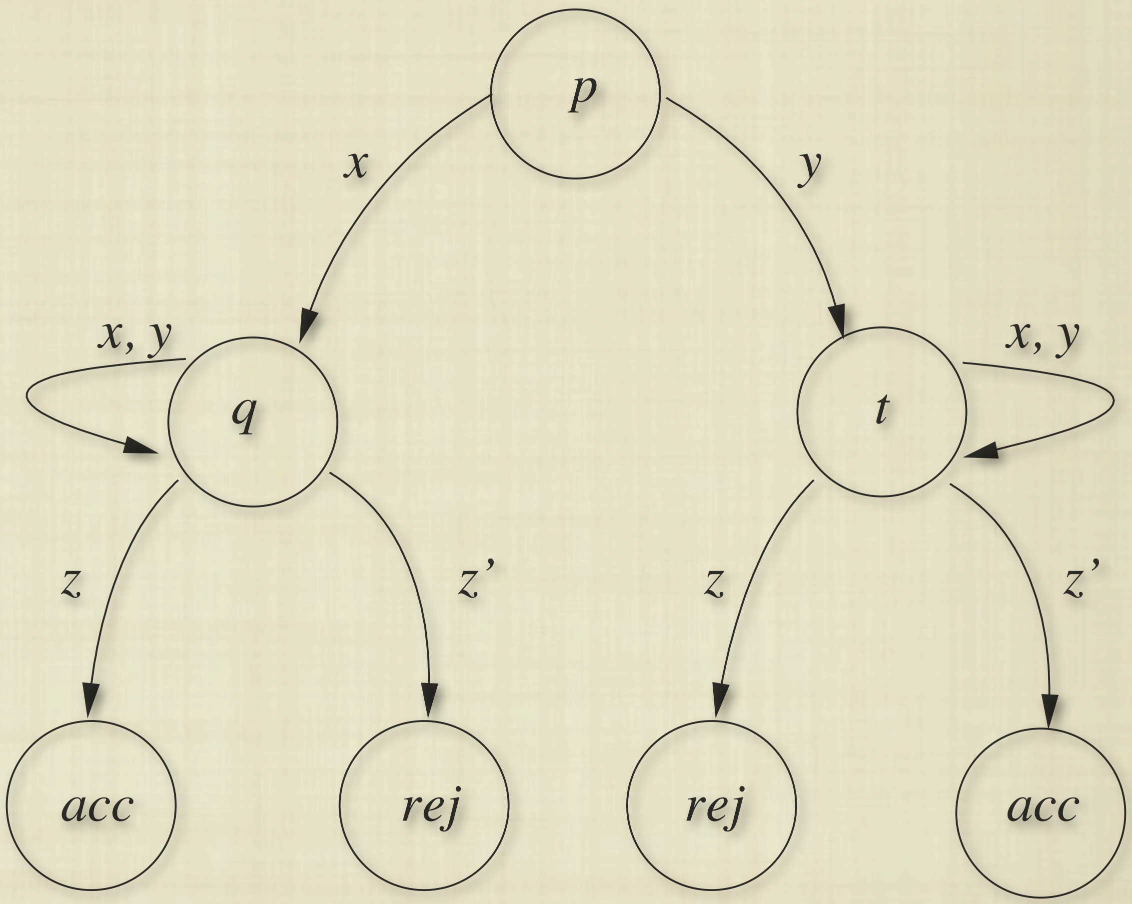


Forbidden constructions

- If a language L contains one of the following patterns in its minimal dfa



Brodsky, Pippenger '99

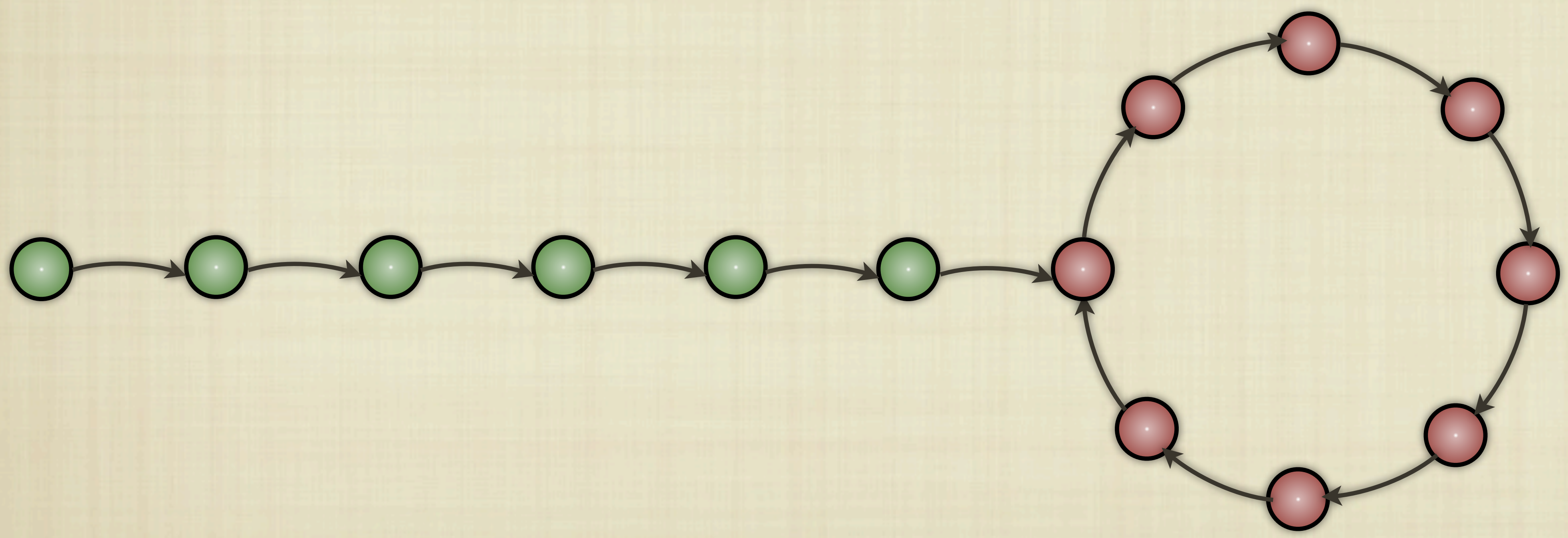


Ambainis, Kikusts, Valdats '00

then it can not be recognized by a MM-qfa.

Unary regular languages

- Unary language: $L \subseteq \{\sigma\}^*$
- Standard dfa for a unary language L:



T transient states

P ergodic states

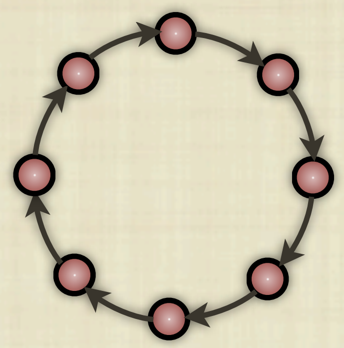
Recognition with qfa's

MM-qfa

T states

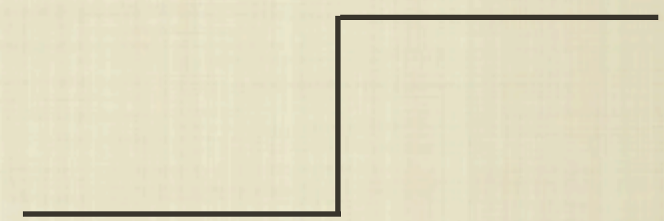
MO-qfa

P states



MM-qfa

T states



$$\frac{1}{2}\,\chi_T(n) \quad + \quad \frac{1}{4}\,\chi_P(n) \quad + \quad \frac{1}{4}\,\chi_{\hat{T}}(n)$$

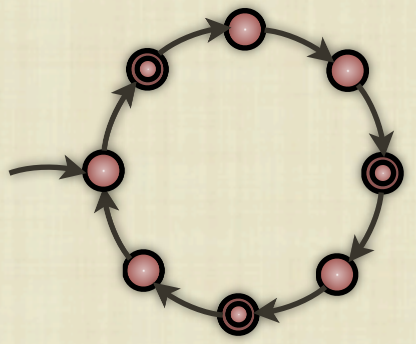
Recognition with cut point $\frac{3}{8}$ isolated by $\frac{1}{8}$, using $O(T+P)$ states

Properties of qfa's

- An event p is m-periodic if, for every $k \geq 0$, it holds

$$p(k) = p(k+m)$$

- Example of P-periodic event: $\chi_P(n)$



- **Theorem 3.** [7] *Let p be an m-periodic event whose discrete Fourier transform $\mathcal{F}(p)$ satisfies $\|\mathcal{F}(p)\|_1 = \sum_{i=1}^{m} |(\mathcal{F}(p))_i| \leq m$. Then, there exists a MO-1qfa A with $O(\frac{\log m}{\delta^2})$ pure states such that p_A is a δ-approximation of the event $\frac{1}{2} + \frac{1}{2}p$.*

Particular cases

- If $\|\mathcal{F}(\chi_P)\|_1 \leq P$:

$$\frac{1}{2}\,\chi_T(n) \quad + \quad \frac{1}{4}\,\chi_P(n) \quad + \quad \frac{1}{4}\,\chi_{\hat{T}}(n)$$

$$\Big\downarrow \text{By Theorem 3}$$

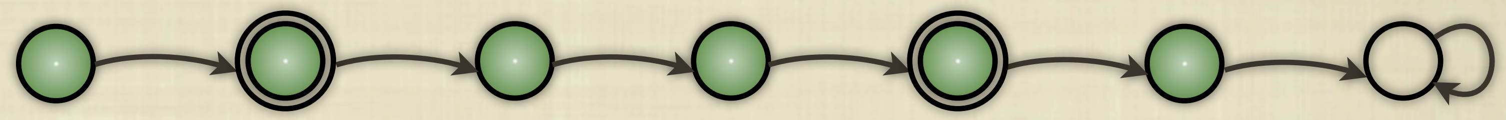
$$\frac{1}{2}\chi_T(n) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\chi_P(n) + f_\delta(n)\right) + \frac{1}{4}\chi_{\hat{T}}(n)$$

Recognition with cut point $\frac{7}{16}$ isolated by $\frac{1}{32}$, using $O(T + \log P)$ states.

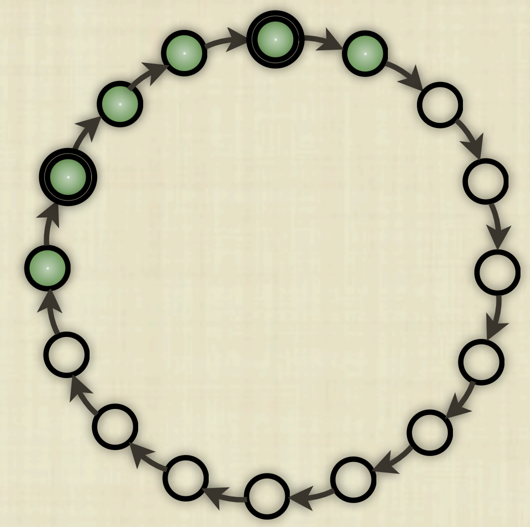
Particular cases

Ultimately periodic languages of period 1:

- $L = L_T$



χ_{T°



T^3 states

If $\|\mathcal{F}(\chi_{T^\circ})\|_1 \le T^3$ we can induce the event

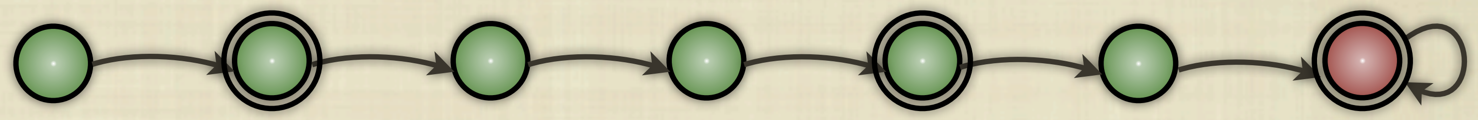
$$\underbrace{\boxed{(\cos\theta)^{2n}}}_{\text{2 states}}\left(\frac{1}{2} + \frac{1}{2}\chi_{T^\circ}(n) + f_{1/8}(n)\right)$$

and recognize L with cut point $\frac{11}{16}$ isolated by $\frac{1}{16}$, using $O(\log T)$ states.

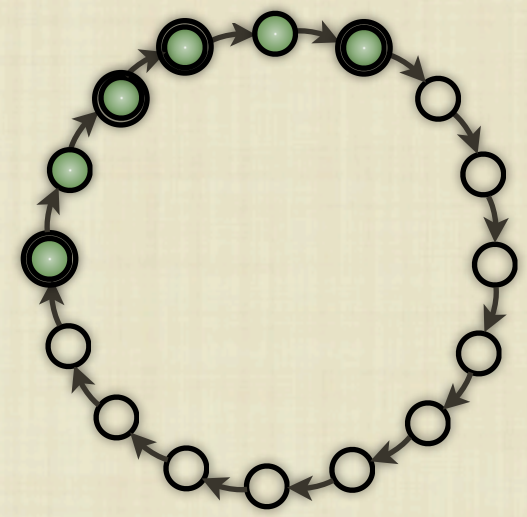
Particular cases

Ultimately periodic languages of period 1:

- $L = L_T \cup \Sigma^{\geq T}$



$\bar{\chi}_{T^\circ}$



T^3 states

Similarly, if $\|\mathcal{F}(\bar{\chi}_{T^\circ})\|_1 \leq T^3$ we can induce the event

$$1 - (\cos\theta)^{2n}\left(\frac{1}{2} + \frac{1}{2}\bar{\chi}_{T^\circ}(n) + f_{1/8}(n)\right)$$

and recognize L with cut point $\frac{5}{16}$ isolated by $\frac{1}{16}$, using $O(\log T)$ states.

Establishing d-periodic behaviors

d – PERIODICITY

- Input: a unary qfa A and an integer $d > 0$.
- Question: is p_A a d-periodic event?

- For any unary MM-qfa we give a simple representation $(\tilde{\pi}, \tilde{U}, \tilde{\eta}_1, \tilde{\eta}_2)$,

such that

$$p_A(n) = \sum_{i=1}^{n-1} \tilde{\pi}\tilde{U}^i\tilde{\eta}_1 + \tilde{\pi}\tilde{U}^n\tilde{\eta}_2$$

- $\tilde{\pi} = \pi \otimes \pi^*$,
- $\tilde{U} = (U(\sigma)P_I(g)) \otimes (U(\sigma)P_I(g))^*$,
- $\tilde{\eta}_1 = (U(\sigma) \otimes U(\sigma)^*) \sum_{j=1}^{m} (P_I(a))_j \otimes (P_I(a))_j$,
- $\tilde{\eta}_2 = (U(\#) \otimes U(\#)^*) \sum_{j=1}^{m} (P_F(a))_j \otimes (P_F(a))_j$.

- Dimension: $m \rightarrow m^2$

Establishing d-periodic behaviors

- d-periodicity condition: $\forall n \in \mathbb{N}\ p_A(n) = p_A(n+d)$

$$\forall n \in \mathbb{N} \quad \sum_{i=1}^{n-1} \tilde{\pi}\tilde{U}^i\tilde{\eta}_1 + \tilde{\pi}\tilde{U}^n\tilde{\eta}_2 = \sum_{i=1}^{n+d-1} \tilde{\pi}\tilde{U}^i\tilde{\eta}_1 + \tilde{\pi}\tilde{U}^{n+d}\tilde{\eta}_2$$

$$\Big\Updownarrow \quad \text{Generating function}$$

$$\tilde{\pi}(I-\tilde{U}^d)\sum_{k=0}^{+\infty}\left(\sum_{i=0}^{k}\tilde{U}^i\right)z^k\tilde{\eta}_1 - \tilde{\pi}\sum_{i=0}^{d-1}\tilde{U}^i\tilde{\eta}_1\sum_{k=0}^{+\infty}z^k = \tilde{\pi}(\tilde{U}^d - I)\sum_{k=0}^{+\infty}(\tilde{U}z)^k\tilde{\eta}_3$$

$$\Big\Updownarrow \quad \text{Properties of matrices}$$

$$\frac{1}{1-z}\tilde{\pi}(I-\tilde{U}^d)(I-\tilde{U}z)^{-1}\tilde{\eta}_1 - \frac{1}{1-z}\tilde{\pi}\sum_{i=0}^{d-1}\tilde{U}^i\tilde{\eta}_1 = \tilde{\pi}(\tilde{U}^d - I)(I-\tilde{U}z)^{-1}\tilde{\eta}_3$$

$$\Big\Updownarrow \quad \text{Adjugate matrix: } A(z) = (\mathrm{adj}(I-\tilde{U}z))^T$$

$$\tilde{\pi}(I-\tilde{U}^d)A(z)\tilde{\eta}_1 - \det(I-\tilde{U}z)\cdot\tilde{\pi}\sum_{i=0}^{d-1}\tilde{U}^i\tilde{\eta}_1 = (1-z)\cdot\tilde{\pi}(\tilde{U}^d - I)A(z)\tilde{\eta}_3$$

Establishing d-periodic behaviors

- d-periodicity condition: $\forall n \in \mathbb{N}\ p_A(n) = p_A(n+d)$

$$\forall n \in \mathbb{N} \quad \sum_{i=1}^{n-1} \tilde{\pi}\tilde{U}^i\tilde{\eta}_1 + \tilde{\pi}\tilde{U}^n\tilde{\eta}_2 = \sum_{i=1}^{n+d-1} \tilde{\pi}\tilde{U}^i\tilde{\eta}_1 + \tilde{\pi}\tilde{U}^{n+d}\tilde{\eta}_2$$

$$\Updownarrow$$

$$\tilde{\pi}(I - \tilde{U}^d)A(z)\tilde{\eta}_1 - \det(I - \tilde{U}z) \cdot \tilde{\pi}\sum_{i=0}^{d-1} \tilde{U}^i\tilde{\eta}_1 = (1-z) \cdot \tilde{\pi}(\tilde{U}^d - I)A(z)\tilde{\eta}_3$$

$$\Updownarrow$$

$$P(z) = P'(z)$$

- In the input qfa has rational entries, so do $P(z)$ and $P'(z)$, and their degree is at most the dimension of $\tilde{U}$, so the check can be done in polynomial time.

Analyzing the non-halting space

- **Lemma.** [Ambainis, Freivalds '98] There exist two subspaces E_1, E_2 such that $E_g = E_1 \oplus E_2$, and
  - if $\pi \in E_1$, then $\pi U(\sigma)P(g) \in E_1$ and $\|\pi U(\sigma)P(g)\| = \|\pi\|$;
  - if $\pi \in E_2$, then $\|\pi(U(\sigma)P(g))^k\| \to 0$, for $k \to \infty$.
- E_1 is the ergodic space and E_2 the transient space.
- Problem: finding the dimension of E_1 (and E_2).
- Key idea: count the modulus 1 eigenvalues of the restriction of $U(\sigma)P(g)$ to E_g.
- For any MM-qfa $A = (\pi, \{U(\sigma), U(\sharp)\}, \mathcal{O})$, there exists an equivalent $A' = (\pi', \{M(\sigma), M(\sharp)\}, \mathcal{O}')$ described by real entries [Blondel et al., '05]

$$\pi_i = a + ib \quad \longrightarrow \quad (a\ ,\ b)$$

$$U_{ij} = c + id \quad \longrightarrow \quad \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$$

Analyzing the non-halting space

Let U_g and M_g be the restriction of, resp., $U(\sigma)$ and $M(\sigma)$ to E_g .

We show that:

- If $\lambda_1, \ldots, \lambda_\mu$ are the eigenvalues of U_g, then the eigenvalues of M_g are $\lambda_1, \ldots, \lambda_\mu, \lambda_1^*, \ldots, \lambda_\mu^*$.
- If $\lambda_j = e^{i\theta_j}$, then $\lambda_j^* = e^{-i\theta_j} = \frac{1}{\lambda_j}$ is a root of $q_{M_g}(\lambda)$;

  if $|\lambda_j| < 1$ then $\left|\frac{1}{\lambda_j}\right| > 1$, therefore $\dfrac{1}{\lambda_j}$ cannot be a root of $q_{M_g}(\lambda)$.

Our algorithm:

- compute $h_{M_g}(\lambda) = \gcd(q_{M_g}(\lambda), q_{M_g}(\frac{1}{\lambda}))$;
- output: $\deg[\, h_{M_g}(\lambda)] \,/\, 2$;

Time complexity:

- polynomial if A has rational entries.

Conclusion

Contributions:

- characterization of the class of languages recognized by unary MM-qfa's;
- families of languages for which the recognizing MM-qfa is exponentially smaller;
- decision problems on d-periodicity;
- analysis of the non-halting subspaces dimensions.

Open problems:

- MM-qfa's for more general classes of languages;
- other periodicity problems on events and languages.