Automorphism Classification of Cellular Automata

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Outline of the talk

- Preliminaries : Cellular Automaton $CA = (\mathbb{Z}^d, Q, f, \nu)$.
- Automorphism of CA is defined by means of a pair of permutations (π, φ) of the neighborhood ν and the state set Q:

$$A \underset{(\pi,\varphi)}{\cong} B \Longleftrightarrow (f_B, \nu_B) = (\varphi^{-1} f_A^{\pi} \varphi, \nu_A^{\pi}).$$

- Classification of local functions $\mathcal{P}_{n,q}$ using permutation group

$$Aut(n, q) \stackrel{\Delta}{=} \{(\pi, \varphi) | \pi \in S_n, \varphi \in S_q\} = S_n \times S_q.$$

- Classification of 256 ELF.
- Group action (X, G), where $X = \mathcal{P}_{n,q}$ and $G = Aut(n, q)$.

Cellular automaton, local structure

Definition

A cellular automaton is defined by a 4-tuple $(\mathbb{Z}^d, Q, f, \nu)$.

- $\mathbb{Z}^d$ is a d-dimensional Euclidean space.
- Q is a finite set of cell states.
- $f : Q^n \to Q$ is a local function in n variables.
- $\nu : \mathbb{N}_n \to \mathbb{Z}^d$ is a neighborhood, where $\mathbb{N}_n = \{1, 2, \ldots, n\}$ and $n \in \mathbb{N}$. This can be seen as a list $\nu = (\nu_1, \ldots, \nu_n)$, where $\nu_i = \nu(i), 1 \leq i \leq n$.

Definition

A pair (f, ν) is called a local structure of CA. We call n the arity of the local structure.

Global function (CA map)

Definition

A local structure uniquely induces a global function $F : Q^{\mathbb{Z}^d} \to Q^{\mathbb{Z}^d}$ defined by

$$F(c)(p) = f(c(p + \nu_1), c(p + \nu_2), ..., c(p + \nu_n)),$$

for any global configuration $c \in Q^{\mathbb{Z}^d}$, where $c(p)$ is the state of cell $p \in \mathbb{Z}^d$ in c.

Reduced local structures

Definition

A local structure is called reduced, if and only if the following conditions are fulfilled:

- f depends on all arguments.
- ν is injective, i.e. $\nu_i \neq \nu_j, i \neq j$ in the list of neighborhood ν.

Remark

In this paper we assume that local structures are reduced, though the theory generalizes to the non-reduced case.

Equivalence of local structures

Definition

Two local structures (f, ν) and (f', ν') are called equivalent, denoted by $(f, \nu) \approx (f', \nu')$, if and only if they induce the same global function.

Lemma

For each local structure (f, ν) there is an equivalent reduced local structure (f', ν').

Permutation of local structures

Definition

Let π denote a permutation of the numbers in $\mathbb{N}_n$.

- For a neighborhood ν, denote by ν^π the neighborhood defined by $\nu^\pi_{\pi(i)} = \nu_i$ for $1 \leq i \leq n$.
- For an n-tuple $\ell \in Q^n$, denote by ℓ^π the permutation of ℓ such that $\ell^\pi(i) = \ell(\pi(i))$ for $1 \leq i \leq n$.
  For a local function $f : Q^n \to Q$, denote by f^π the local function $f^\pi : Q^n \to Q$ such that $f^\pi(\ell) = f(\ell^\pi)$ for all ℓ.

Example

Symmetric group $S_3 = \{\pi_i, 0 \leq i \leq 5\}$.

$$\pi_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \pi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \pi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

6 Permutations of the elementary neighborhood ENB $(-1, 0, 1)$ are isomorphic to S_3.

$$ENB^{\pi_0} = (-1, 0, 1), ENB^{\pi_1} = (-1, 1, 0), ENB^{\pi_2} = (0, -1, 1),$$

$$ENB^{\pi_3} = (0, 1, -1), ENB^{\pi_4} = (1, -1, 0), ENB^{\pi_5} = (1, 0, -1)$$

Lemma

(f, ν) and (f^π, ν^π) are equivalent for any permutation π.

Lemma

If (f, ν) and (f', ν') are two equivalent reduced local structures, then there is a permutation π such that $\nu^\pi = \nu'$.

Theorem

If (f, ν) and (f', ν') are two reduced local structures which are equivalent, then there is a permutation π such that $(f^\pi, \nu^\pi) = (f', \nu')$.

Polynomials over finite fields

Q is a finite field $GF(q)$ and $f : Q^n \rightarrow Q$ is a polynomial over $GF(q)$ in n indeterminates $x_1, ..., x_n$ of degree less than q in each indeterminate. The set of such polynomials is denoted by $\mathcal{P}_{n,q},\ n \geq 1, q \geq 2$.

If $f \in \mathcal{P}_{3,q}$,

$$f(x_1, x_2, x_3) = u_0 + u_1 x_1 + u_2 x_2 + \cdots + u_i x_1^h x_2^j x_3^k + \cdots \\ + u_{q^3-2} x_1^{q-1} x_2^{q-1} x_3^{q-2} + u_{q^3-1} x_1^{q-1} x_2^{q-1} x_3^{q-1}, \\ \text{where } u_i \in GF(q),\ 0 \leq i \leq q^3 - 1. \quad (1)$$

If $f \in \mathcal{P}_{3,2}$ (Boolean function),

$$f(x_1, x_2, x_3) = u_0 + u_1 x_1 + u_2 x_2 + u_3 x_3 \\ + u_4 x_1 x_2 + u_5 x_1 x_3 + u_6 x_2 x_3 + u_7 x_1 x_2 x_3, \\ \text{where } u_i \in GF(2) = \{0, 1\}, 0 \leq i \leq 7. \quad (2)$$

Note that $a \vee b$ (Boolean) $= a + b + ab$ (polynomial), $a \wedge b = ab$.

A conclusion

Summing up the above discussions, we have the following corollary, which gives a reason why we only consider the set of local functions when classifying CA.

Corollary

As far as the equivalence of CA (and the automorphism classification thereof) is concerned, we only have to consider the local functions without explicitly referring to neighborhoods.

Automorphism

Assume that $A = (\mathbb{Z}^d, Q, f_A, \nu_A)$ and $B = (\mathbb{Z}^d, Q, f_B, \nu_B)$ are two CA having the same arity of local structures. Now we consider a pair of permutations (π, φ), where π and φ are permutations of ν and Q, respectively. Note that φ naturally extends to $\varphi : Q^{\mathbb{Z}^d} \to Q^{\mathbb{Z}^d}$.

Definition

Two CA A and B are called automorphic, denoted $A \cong B$, if and only if there is a pair of permutations (π, φ) such that

$$(f_B, \nu_B) = (\varphi^{-1} f_A^{\pi} \varphi, \nu_A^{\pi}).$$

In this case, (π, φ) is called an automorphism of CA.
Symbolically we write $A \underset{(\pi,\varphi)}{\cong} B$.

Example

ECA : $Q = GF(2) = \{0,1\}$. ELF : $Q^3 \to Q$. ENB=$(-1,0,1)$.
The permutation (conjugation) of states $0 \leftrightarrow 1$.
$f'(x_1,...,x_n) = \varphi_1^{-1} f \varphi_1 = 1 + f(1+x_1,...,1+x_n)$.

- Universal function $f_{110} = x_1x_2x_3 + x_2x_3 + x_2 + x_3$.
$$f_{110}^{\pi_2} = f_{122} = x_1x_2x_3 + x_1x_3 + x_1 + x_3.$$
$(f_{110}, ENB) \neq (f_{122}, ENB)$, but $(f_{110}^{\pi_2}, ENB^{\pi_2}) = (f_{122}, ENB)$ or $(f_{110}, ENB) \underset{(\pi_2,\varphi_0)}{\cong} (f_{122}, ENB)$.
- By π_5 and conjugation φ_1, we see $(f_{110}, ENB) \underset{(\pi_5,\varphi_1)}{\cong} (f_{193}, ENB)$.
Thus we have $(f_{110}, ENB) \cong (f_{122}, ENB) \cong (f_{193}, ENB)$.
- In total there are 6 ECA which are automorphic to (f_{110}, ENB).

Automorphism group of CA

We see that the sets of all permutations π of ν and φ of Q are isomorphic to symmetric groups S_n and S_q, respectively. Then we have

Definition

$$Aut(n,q) \equiv \{(\pi,\varphi)|\pi \in S_n, \varphi \in S_q\} \sim S_n \times S_q. \tag{3}$$

$Aut(n,q)$ will be called an automorphism group of CA. Note that since symmetric groups are generally nonabelian, $Aut(n,q)$ is nonabelian.

Automorphism classification of CA

Lemma

Automorphism group $Aut(n, q)$ naturally induces a classification of local structures of CA.

Proof: Let A, B and C be local structures of CA. Then we see that if $A \underset{(\pi,\varphi)}{\cong} B$ and $B \underset{(\pi',\varphi')}{\cong} C$ for some $\pi, \pi' \in S_n$ and $\varphi, \varphi' \in S_q$, then $A \underset{(\pi'\pi,\varphi'\varphi)}{\cong} C$. It is seen that the relation $\underset{(\pi,\varphi)}{\cong}$ is an equivalence relation which induces a classification of CA. ■

Definition

The classification induced by $Aut(n, q)$ is called an automorphism classification of $\mathcal{P}_{n,q}$ denoted $\mathcal{NW} : \{[f_1], [f_2], ..., [f_m]\}$, where f_i is a representative of class $[f_i], 1 \le i \le m$. m will be called the size of automorphism classification.
In other words, $f' \in [f]$ if and only if there is a $(\varphi, \pi) \in Aut(n, q)$ such that $(f', \nu') = (\varphi^{-1} f^{\pi} \varphi, \nu^{\pi})$.

Remark

All CA that have the local functions from a class provide the same global properties like surjectivity, injectivity and reversibility, provided that the local structures are permuted appropriately. In this sense we say that CA have a certain property up to permutations.

Example

NW 9 ($|[f_{10}]| = 12$)

$f_{10} = x_3 + x_1x_3$. $f_{10}^{\pi_1} = x_2 + x_1x_2 = f_{12}$.

$f'_{10} = 1 + x_1 + x_1x_3 = f_{175}$. $f'^{\pi_1}_{10} = 1 + x_1 + x_1x_2 = f_{207}$.

Wolfram number

$\varphi \setminus \pi$	π_0	π_1	π_2	π_3	π_4	π_5
φ_0	f_{10}	f_{12}	f_{34}	f_{68}	f_{48}	f_{80}
φ_1	f_{175}	f_{207}	f_{187}	f_{221}	f_{243}	f_{245}

Polynomial

$\varphi \setminus \pi$	π_0	π_1	π_2	π_3	π_4	π_5
φ_0	$x_3 + x_1x_3$	$x_2 + x_1x_2$	$x_3 + x_2x_3$	$x_2 + x_2x_3$	$x_1 + x_1x_2$	$x_1 + x_1x_3$
φ_1	$1 + x_1 + x_1x_3$	$1 + x_1 + x_1x_2$	$1 + x_2 + x_2x_3$	$1 + x_3 + x_2x_3$	$1 + x_2 + x_1x_2$	$1 + x_3 + x_1x_3$

Example

NW 32 ($|[f_{110}]| = 6$)

$f_{110} = x_1 x_2 x_3 + x_2 x_3 + x_2 + x_3.$ (computation universal)

$f'_{110} = x_1 x_2 x_3 + x_1 x_2 + x_1 x_3 + x_1 + x_2 + x_3 + 1 = f_{137}.$

$\varphi \setminus \pi$	π_0, π_1	π_2, π_4	π_3, π_5
φ_0	f_{110}	f_{122}	f_{124}
φ_1	f_{137}	f_{161}	f_{193}

Example

NW 40 ($|[f_{150}]| = 1$)

$f_{150} = x_1 + x_2 + x_3$. (symmetric function)

$f'_{150} = x_1 + x_2 + x_3 = f_{150}$.

$\varphi \setminus \pi$	$\pi_0, \pi_1, \pi_2, \pi_4, \pi_3, \pi_5$
φ_0	f_{150}
φ_1	f_{150}

Automorphism classification of ELF

In Table 1 the 256 Elementary Local Functions (ELF) $f_i, \; 0 \leq i \leq 255$ in Wolfram numbers are classified into 46 automorphism classes $NWi, \; 1 \leq i \leq 46$.

The 7 classes indexed by * are surjective but not injecitve up to permutations.

6 functions in NW12** and NW44** are injective and surjective, i.e. reversible.

The other classes are neither surjective nor injective.

6 functions in NW32+ are automorphic to the universal function f_{110}.

In Table 1, every class is indexed by NW i, $1 \leq i \leq 46$. Conjugate functions are bracketed, where singletons are self-conjugate functions.

Table 1-1. Automorphism classification of ELF

NW	Automorphism classes
1	$\{f_0, f_{255}\}$
2	$\{f_1, f_{127}\}$
3	$\{f_2, f_{191}\} \cup \{f_{16}, f_{247}\} \cup \{f_4, f_{223}\}$
4	$\{f_3, f_{63}\} \cup \{f_{17}, f_{119}\} \cup \{f_5, f_{95}\}$
5	$\{f_6, f_{159}\} \cup \{f_{20}, f_{215}\} \cup \{f_{18}, f_{183}\}$
6	$\{f_7, f_{31}\} \cup \{f_{21}, f_{87}\} \cup \{f_{19}, f_{55}\}$
7	$\{f_8, f_{239}\} \cup \{f_{64}, f_{253}\} \cup \{f_{32}, f_{251}\}$
8	$\{f_9, f_{111}\} \cup \{f_{65}, f_{125}\} \cup \{f_{33}, f_{123}\}$
9	$\{f_{10}, f_{175}\} \cup \{f_{80}, f_{245}\} \cup \{f_{12}, f_{207}\} \cup \{f_{68}, f_{221}\} \cup \{f_{34}, f_{187}\} \cup \{f_{48}, f_{243}\}$
10	$\{f_{11}, f_{47}\} \cup \{f_{81}, f_{117}\} \cup \{f_{13}, f_{79}\} \cup \{f_{69}, f_{93}\} \cup \{f_{35}, f_{59}\} \cup \{f_{49}, f_{115}\}$

(continued)

Table 1-2.

NW	automorphism classes
11	$\{f_{14}, f_{143}\} \cup \{f_{84}, f_{213}\} \cup \{f_{50}, f_{179}\}$
12**	$\{f_{15}\} \cup \{f_{51}\} \cup \{f_{85}\}$ (Reversible class)
13	$\{f_{22}, f_{151}\}$
14	$\{f_{23}\}$
15	$\{f_{24}, f_{231}\} \cup \{f_{66}, f_{189}\} \cup \{f_{36}, f_{219}\}$
16	$\{f_{25}, f_{103}\} \cup \{f_{61}, f_{67}\} \cup \{f_{37}, f_{91}\}$
17	$\{f_{26}, f_{167}\} \cup \{f_{82}, f_{181}\} \cup \{f_{28}, f_{199}\} \cup \{f_{70}, f_{157},\} \cup \{f_{38}, f_{155}\} \cup \{f_{52}, f_{211}\}$
18	$\{f_{27}, f_{39}\} \cup \{f_{53}, f_{83}\} \cup \{f_{29}, f_{71}\}$
19*	$\{f_{30}, f_{135}\} \cup \{f_{86}, f_{149}\} \cup \{f_{54}, f_{147}\}$
20	$\{f_{40}, f_{235}\} \cup \{f_{96}, f_{249}\} \cup \{f_{72}, f_{237}\}$

(continued)

Table 1-3.

NW	automorphism classes
21	$\{f_{41}, f_{107}\} \cup \{f_{97}, f_{121}\} \cup \{f_{73}, f_{109}\}$
22	$\{f_{42}, f_{171}\} \cup \{f_{112}, f_{241}\} \cup \{f_{76}, f_{205}\}$
23	$\{f_{43}\} \cup \{f_{77}\} \cup \{f_{113}\}$
24	$\{f_{44}, f_{203}\} \cup \{f_{100}, f_{217}\} \cup \{f_{56}, f_{227}\} \cup \{f_{98}, f_{185}\} \cup \{f_{74}, f_{173}\} \cup \{f_{88}, f_{229}\}$
25*	$\{f_{45}, f_{75}\} \cup \{f_{101}, f_{89}\} \cup \{f_{57}, f_{99}\}$
26	$\{f_{46}, f_{139}\} \cup \{f_{116}, f_{209}\} \cup \{f_{58}, f_{163}\} \cup \{f_{114}, , f_{177}\} \cup \{f_{78}, f_{141}\} \cup \{f_{92}, f_{197}\}$
27*	$\{f_{60}, f_{195}\} \cup \{f_{102}, f_{153}, \} \cup \{f_{90}, f_{165}\}$
28	$\{f_{62}, f_{131}\} \cup \{f_{118}, f_{145}\} \cup \{f_{94}, f_{133}\}$
29	$\{f_{104}, f_{233}\}$
30*	$\{f_{105}\}$

(continued)

Table 1-4.

NW	Automorphism classes
31*	$\{f_{106}, f_{169}\} \cup \{f_{120}, f_{225}\} \cup \{f_{108}, f_{201}\}$
32+	$\{f_{110}, f_{137}\} \cup \{f_{124}, f_{193}\} \cup \{f_{122}, f_{161}\}$ (Universal class)
33	$\{f_{126}, f_{129}\}$
34	$\{f_{128}, f_{254}\}$
35	$\{f_{130}, f_{190}\} \cup \{f_{144}, f_{246}\} \cup \{f_{132}, f_{222}\}$
36	$\{f_{134}, f_{158}\} \cup \{f_{148}, f_{214}\} \cup \{f_{146}, f_{182}\}$
37	$\{f_{136}, f_{238}\} \cup \{f_{192}, f_{252}\} \cup \{f_{160}, f_{250}\}$
38	$\{f_{138}, f_{174}\} \cup \{f_{208}, f_{244}\} \cup \{f_{140}, f_{206}\} \cup \{f_{196}, f_{220}\} \cup \{f_{162}, f_{186}\}$ $\cup\{f_{176}, f_{242}\}$
39	$\{f_{142}\} \cup \{f_{212}\} \cup \{f_{178}\}$
40*	$\{f_{150}\}$
41	$\{f_{152}, f_{230}\} \cup \{f_{194}, f_{188},\} \cup \{f_{164}, f_{218}\}$
42*	$\{f_{154}, f_{166}\} \cup \{f_{180}, f_{210}\} \cup \{f_{156}, f_{198}\}$
43	$\{f_{168}, f_{234}\} \cup \{f_{224}, f_{248}\} \cup \{f_{200}, f_{236}\}$
44**	$\{f_{170}\} \cup \{f_{240}\} \cup \{f_{204}\}$ (Reversible class)
45	$\{f_{172}, f_{202}\} \cup \{f_{216}, f_{228} \cup \{f_{184}, f_{226}\}$
46	$\{f_{232}\}$

Table 2: Taxonomy of automorphism classification of ELF

number of functions in NW class	number of NW classes	number of functions
12	6	72
6	26	156
3	4	12
2	6	12
1	4	4
total	46	256

Group action

Classification $\mathcal{NW}$ is reformulated as a group action (G, X), where $X = \mathcal{P}_{n,q}$ and $G = Aut(n, q) \sim S_n \times S_q$. X is called a G-space.

- For any $f \in \mathcal{P}_{n,q}$ the automorphism class $[f]$ is now the same as the orbit containing f: $\{gf|g \in G\} = \{\varphi^{-1} f^{\pi} \varphi | \pi \in S_n, \varphi \in S_q\}$.
- A G-space is called transitive if it has just one orbit. Every G-space is a disjoint union of transitive G-spaces. The set of such transitive G-spaces will be called an orbit space or quotient space denoted X/G.
- Every automorphism class $[f]$ is a transitive G-space such that $\mathcal{P}_{n,q} = \cup_{i=1}^{m} [f_i]$. The size of classification m is equal to the number of orbits $|X/G|$ given by the Orbit-Counting Lemma.

Lemma (Orbit-Counting Lemma)

The number of orbits $|X/G|$ is equal to the "average number" of fixed elements in X of an element of G. That is, if $X(g) = |\{x \in X | gx = x\}|$, then we have

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} X(g).$$

Example

For $X = \mathcal{P}_{3,2}$ and $G \approx S_3 \times S_2$, we see that $|X| = 2^{2^3} = 256$ and $|G| = 3! \times 2! = 12$. The orbit number $|X/G| = 46$.
Table 2 in Appendix shows that
$\sum(\text{orbit length} \times \text{number of orbits}) = 12 \times 6 + 6 \times 26 + 3 \times 4 + 2 \times 6 + 1 \times 4$
$= 256$.

Lemma (Lagrange's Theorem)

Let Ω be an arbitrary transitive G-space, then

$$|G| = |\Omega| \cdot |G_x|, \text{where } G_x = \{g \in G | gx = x\}, \forall x \in \Omega.$$

$G_x = \{g \in G | gx = x\}$ *is called the* ***stabilizer*** *of* x.

Example

Lagrange's Theorem applies to each NW class $NW_i \subset \mathcal{P}_{3,2}$. For instance, in case of NW_9 the cardinality of stabilizer $|G_x| = 1$ for any $x \in NW_9$. Therefore we have $|NW_9| = 12$. In contrast we see $G_x = G$ or $|G_x| = 12$ for all $x \in NW_{30}$ and therefore $|NW_{30}| = 1$.

Classification by subgroups of $Aut(n, q)$

Let T_n and T_q be subgroups of S_n and S_q, respectively. Then we can likewise define a classification of $\mathcal{P}_{n,q}$ by $T_n \times T_q$.

Lemma

A smaller subgroup induces a classification with a larger size.

Example

The historical classification of ECA into 88 classes was made by Hurd (1986), which appears in a book by Wolfram (1994), considering the left-right symmetry of ENB and the state conjugation. This classification is induced by a subgroup

$$\{(\pi_0, \varphi_0), (\pi_0, \varphi_1), (\pi_5, \varphi_0), (\pi_5, \varphi_1)\} = \{\pi_0, \pi_5\} \times S_2 \subset S_3 \times S_2$$

Example

Multiplication table of S_3.

$\cdot$	π_0	π_1	π_2	π_3	π_4	π_5
π_0	π_0	π_1	π_2	π_3	π_4	π_5
π_1	π_1	π_0	π_4	π_5	π_2	π_3
π_2	π_2	π_3	π_0	π_1	π_5	π_4
π_3	π_3	π_2	π_5	π_4	π_0	π_1
π_4	π_4	π_5	π_1	π_0	π_3	π_2
π_5	π_5	π_4	π_3	π_2	π_1	π_0

Example

Let $< a, b, .. >$ denote the subgroup generated by subset $\{a, b, ..\}$. Then we see the following.

- $< \pi_0, \pi_1 >= \{\pi_0, \pi_1\}$ (subgroup of right-center symmetry)
- $< \pi_0, \pi_2 >= \{\pi_0, \pi_2\}$ (subgroup of left-center symmetry)
- $< \pi_0, \pi_5 >= \{\pi_0, \pi_5\}$ (subgroup of right-left symmetry)
- $< \pi_0, \pi_3 >=< \pi_0, \pi_4 >=< \pi_0, \pi_3, \pi_4 >= \{\pi_0, \pi_3, \pi_4\}$ (subgroup of cyclic permutations)
- $\{\pi_0, \pi_1, \pi_2\} \subsetneq < \pi_0, \pi_1, \pi_2 >= S_3$
- $\{\pi_0, \pi_1, \pi_5\} \subsetneq < \pi_0, \pi_1, \pi_5 >= S_3$
- $\{\pi_0, \pi_2, \pi_5\} \subsetneq < \pi_0, \pi_2, \pi_5 >= S_3$

Future problems

Problem

Closer view of group action of $S_n \times S_q$ on $\mathcal{P}_{n,q}$ taking advantage of their specific algebraic structures.

We will not make mathematics but contribute some thing to it as well as to the CA study.

Problem

Classification of CA by weaker properties than the sameness of global functions.

- Classification by cognate $\asymp$. For $A = (f, \nu)$ and $B = (f', \nu')$,

$$A \asymp B \Longleftrightarrow (f', \nu') = (f^{\pi}, \nu^{\pi'}), \text{where } \pi \neq \pi'.$$

  Are there interesting and useful properties which are invariant by classification $\asymp$?

- If CA are not cognate or the neighborhoods are not a permutation of each other, we would have infinitely many CA including a computation universal CA.
  Is there an effective classification of such arbitrary set of CA?

Thank you for your attention!