Equivalence relations of Mealy automata

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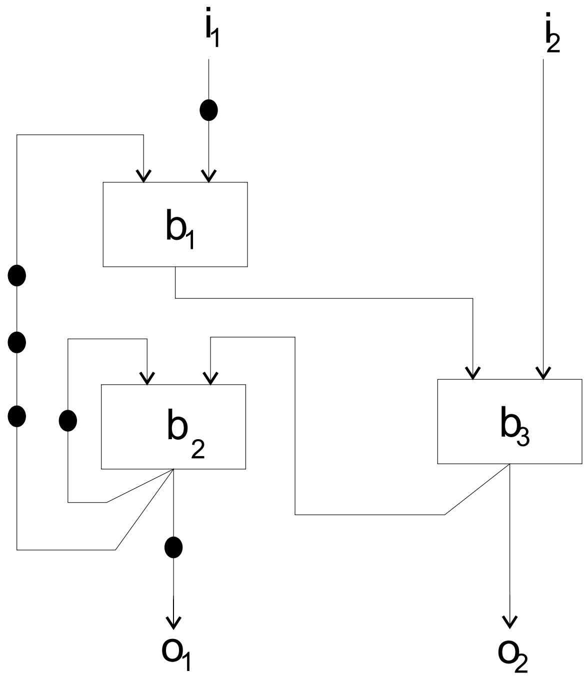
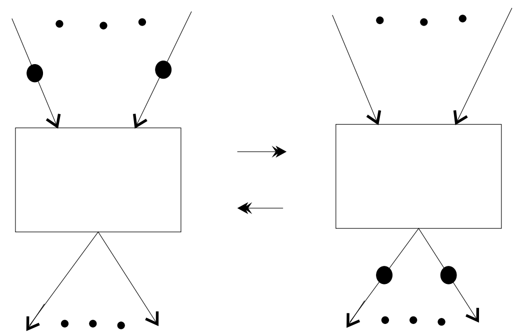
- Motivation
- Categorical background
- Retiming equivalence
- Retiming equivalence in terms of transition diagrams
- Simulation equivalence
- The coincidence of retiming and simulation equivalence

A synchronous system:



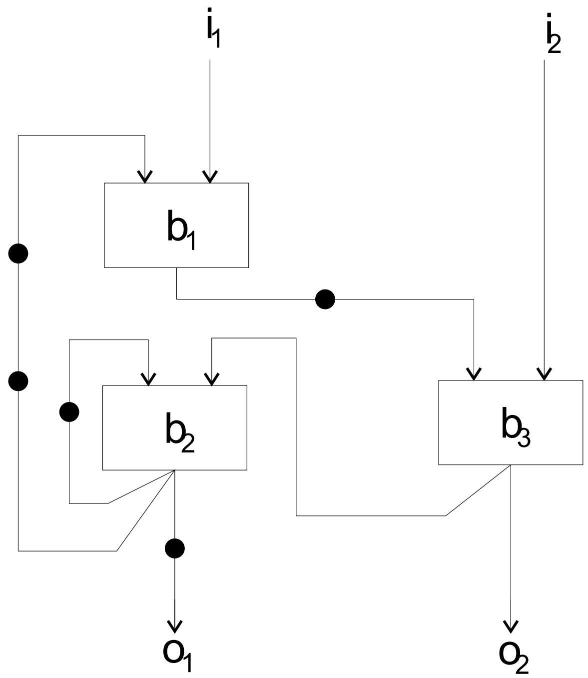
A system is *systolic* if there is at least one register on every interconnection between two functional elements.

Retiming a functional element (box):



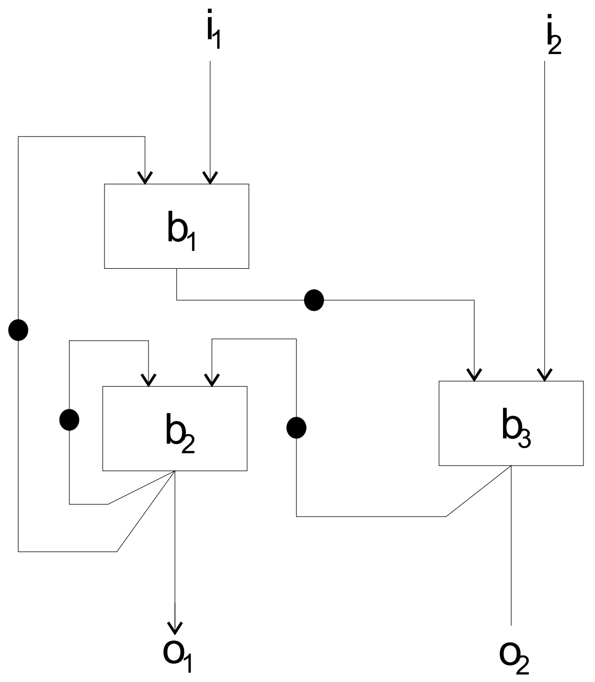
One layer of registers is moved from the input side of the box to the output side (positive retiming) or vice versa (negative retiming).

Retiming (positively) box b_1 in our example system:



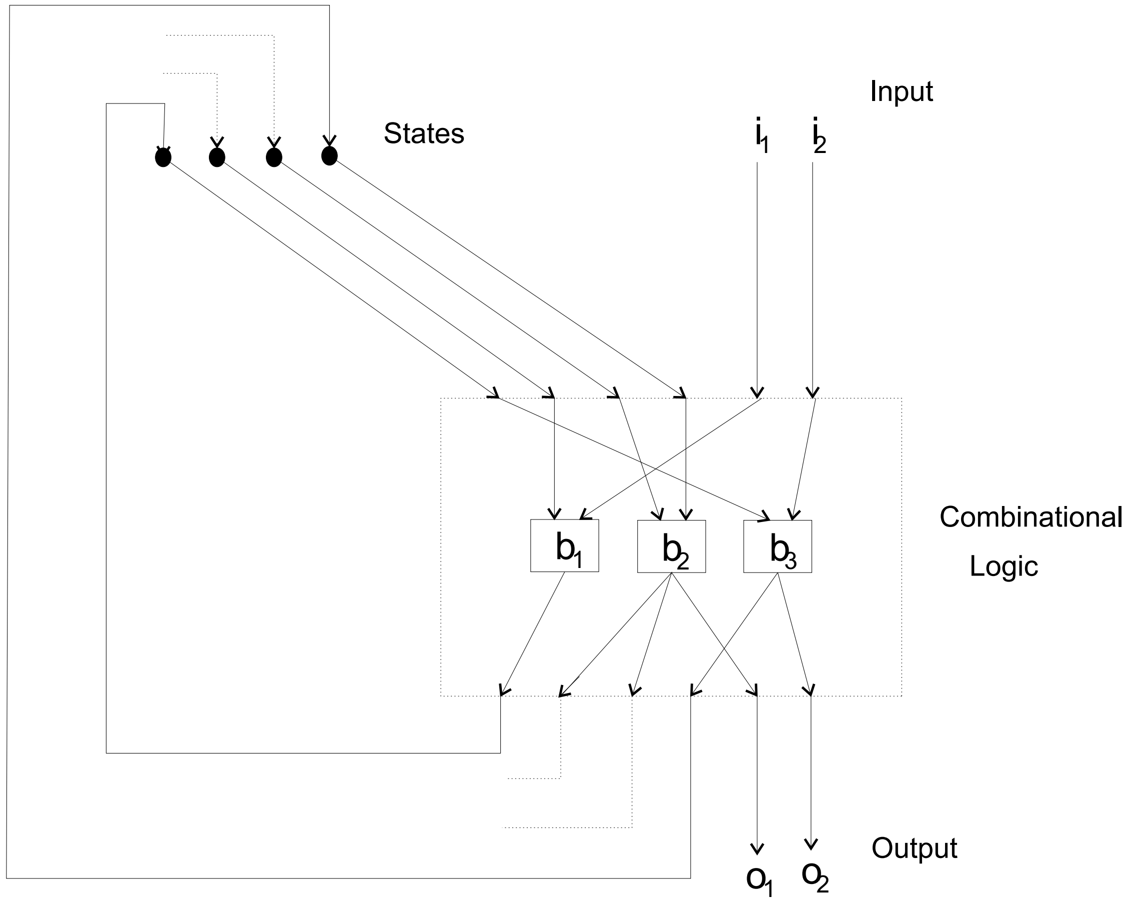
After the retiming, the system is still not systolic.

Retiming (negatively) box b_2:

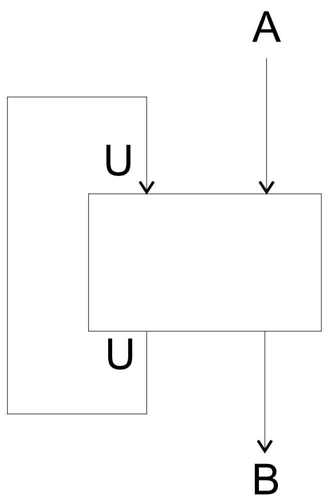


The resulting system is already systolic.
Question: What is the impact of retiming on the behavior of the system?

The system as an automaton:



In general, an automaton "from A to B" is represented by the following diagram



Automaton $(U, \alpha) : A \longrightarrow B$

$(U, \alpha) = \uparrow^U \alpha$, where $\alpha : U \otimes A \rightarrow U \otimes B$

Strict monoidal category

Objects: structured as a monoid equipped with an associative binary operation $\otimes$ and unit object I.

In the category $\boldsymbol{Set}$, $\otimes$ is Cartesian product and $I = \{\emptyset\}$.

Morphisms: if $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$, then

$$f_1 \otimes f_2 : A_1 \otimes A_2 \to B_1 \otimes B_2.$$

Laws:

$$f \otimes 1_I = 1_I \otimes f = f$$

$$(f_1 \otimes f_2) \circ (g_1 \otimes g_2) = (f_1 \circ g_1) \otimes (f_2 \circ g_2)$$

Pictorially, a morphism $f : A \longrightarrow B$ is a box

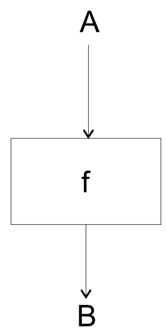
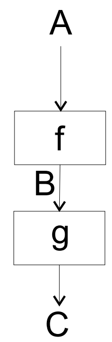
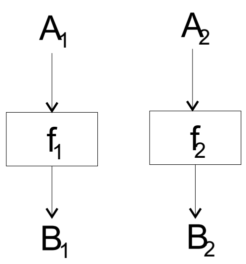


Diagram representation of the monoidal operations:

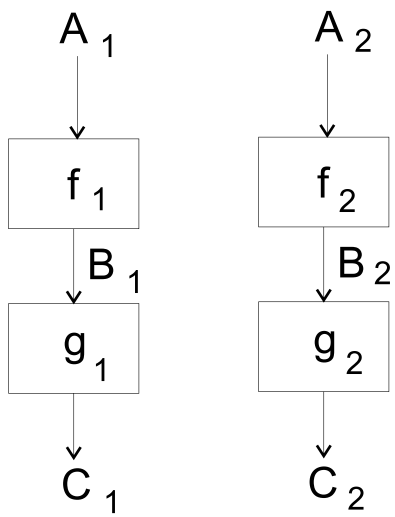


$f \circ g$



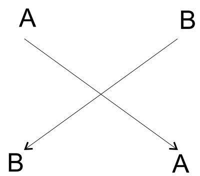
$f_1 \otimes f_2$

The “monoidal” law then manifests itself in the diagram:

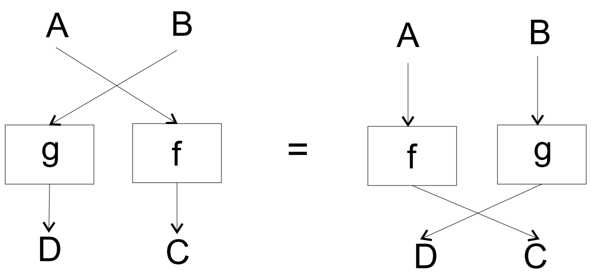


A symmetry in a strict monoidal category is a natural isomorphism

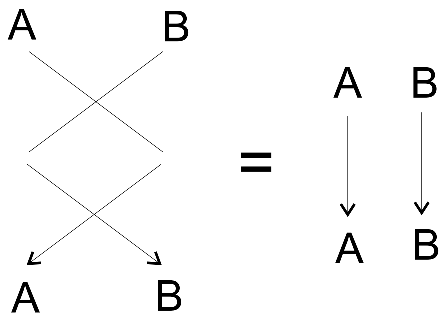
$$\pi_{A,B} : A \otimes B \to B \otimes A$$

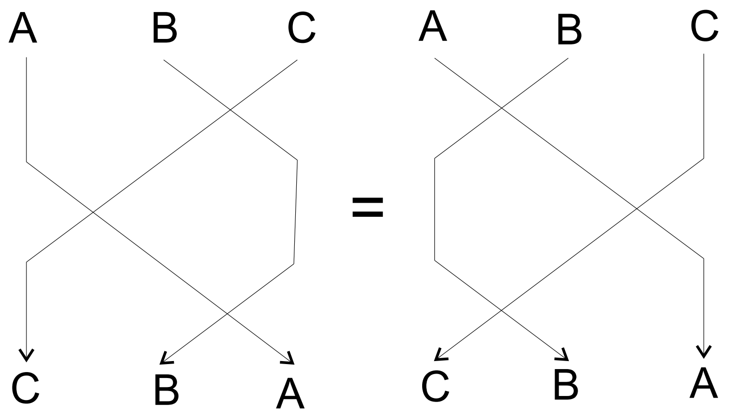


Naturality of symmetry means the following identity:



Symmetry laws:





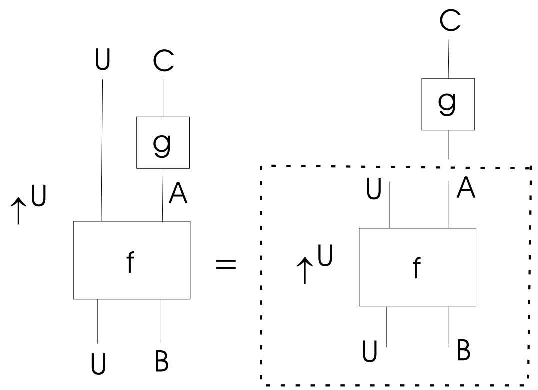
Monoidal category with feedback:

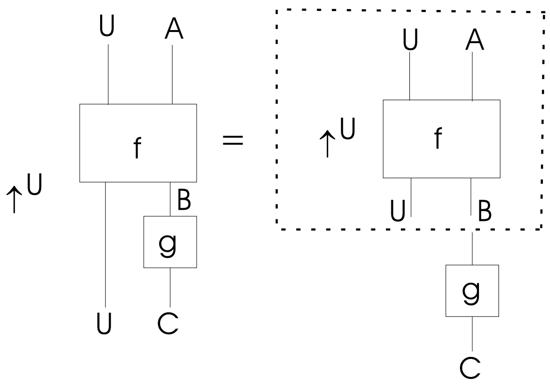
Symmetric strict monoidal category enriched with a feedback operation

$$\uparrow^U f : A \to B, \quad \text{where} \quad f : U \otimes A \to U \otimes B$$

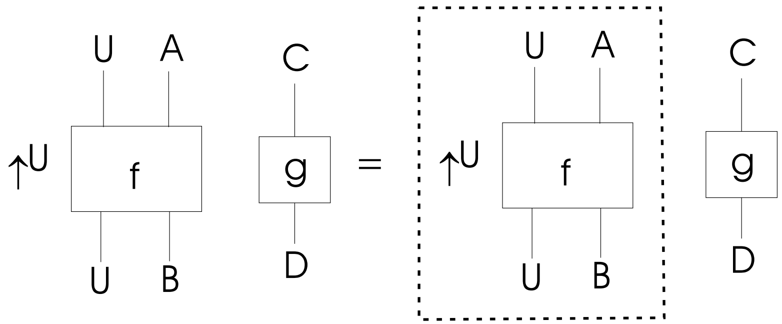
Feedback must obey the "diagram" laws.

Naturality:





Superposing:



Vanishing:

$$\uparrow^{I} f = f\,; \quad \uparrow^{U\otimes V} f = \uparrow^{V} (\uparrow^{U} f).$$

Turning a symmetric monoidal category $\mathcal{M}$ into one with feedback.

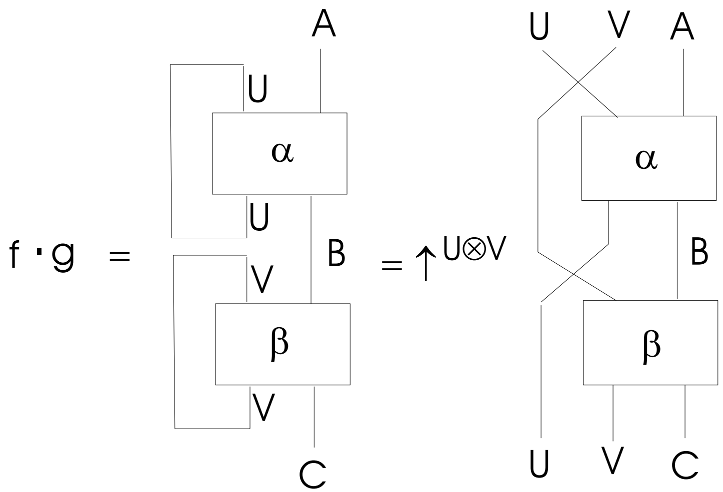
First construct the category $Aut_{\mathcal{M}}$ of automata over $\mathcal{M}$.

Objects: those of $\mathcal{M}$.

Morphisms: pairs (U, α), where $\alpha : U \otimes A \to U \otimes B$.
The pair (U, α) stands for the formal expression $\uparrow^U \alpha$.

Composition in $Aut_{\mathcal{M}}$:

for $f = (U, \alpha) : A \to B$ and $g = (V, \beta) : B \to C$

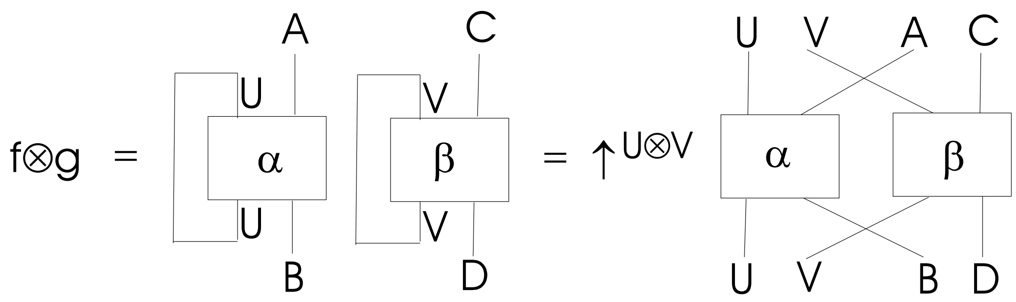


Identities $\mathbf{1}_A : A \to A$ in $Aut_{\mathcal{M}}$:

$(I, (1_A)_{\mathcal{M}})$.

Tensor of automata in $Aut_{\mathcal{M}}$:

for $f = (U, \alpha) : A \to B$ and $g = (V, \beta) : C \to D$



Feedback in $Aut_{\mathcal{M}}$:

If $f = (U, \alpha) : V \otimes A \to V \otimes B$, then

$$\uparrow^V f = (U \otimes V, \alpha) : A \to B.$$

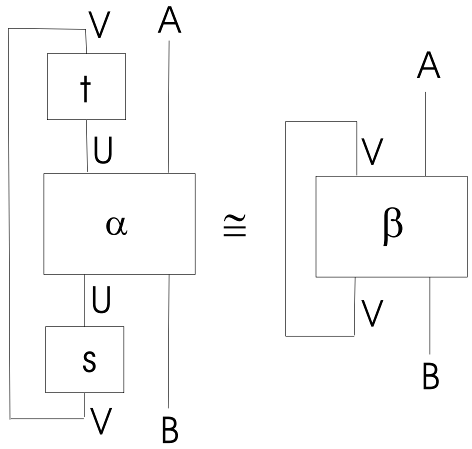
Automata $f = (U, \alpha) : A \to B$ and $g = (V, \beta) : A \to B$

are isomorphic if there exists a pair of isomorphisms

$s : U \to V$ and $t : V \to U$ in $\mathcal{M}$ such that

$$(t \otimes 1_A) \circ \alpha \circ (s \otimes 1_B) = \beta.$$

Isomorphism of automata:

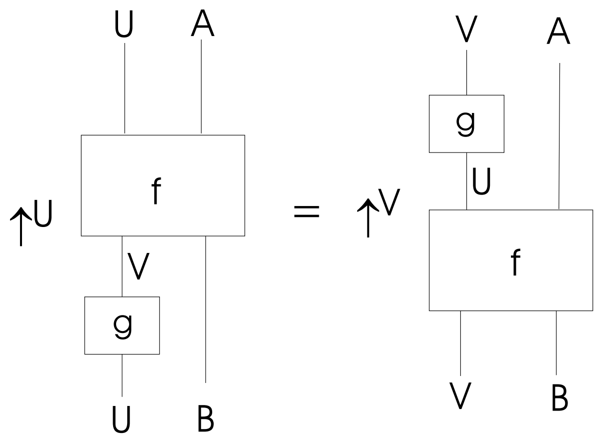


Theorem: (Katis, Sabadini, and Walters, 1997)

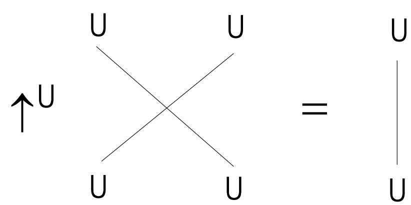
The quotient of the category $Aut_{\mathcal{M}}$ by isomorphism forms a monoidal category $Circ(\mathcal{M})$ with feedback.

Axioms *not* valid in $Circ(\mathcal{M})$

Sliding (circular feedback, retiming):



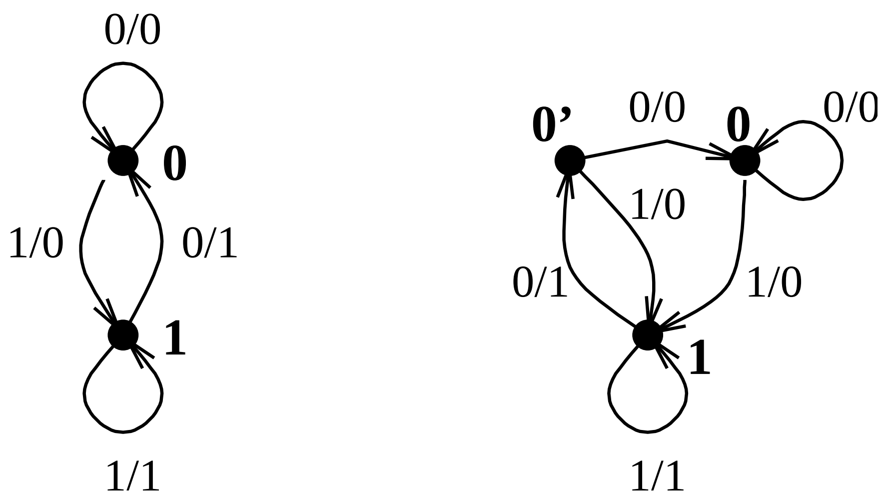
Yanking:



The congruence induced by the sliding axiom in $Circ(\mathcal{M})$ is called *retiming equivalence*.

Finite state deterministic Mealy automata are associated with the choice $\mathcal{M} = \boldsymbol{Set_f}$, the category of finite sets and functions. It is relatively easy to characterize retiming equivalence in this special case, using transition diagrams as a means of comparison.

Example



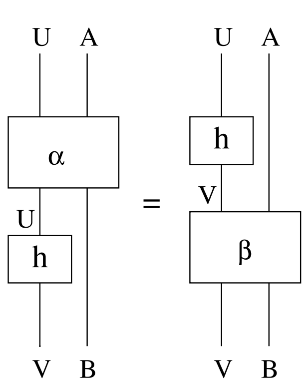
A state of a finite state automaton (U, α) is called *run-out* if it can only be reached by an input string of a bounded length from any state. *Permanent* states are those that are not run-out.

Two states $u, u' \in U$ are said to be *retiming equivalent* if u and u' are equivalent in the usual sense, and, furthermore, u and u' are taken to the same state by α on every sufficiently long input string w.

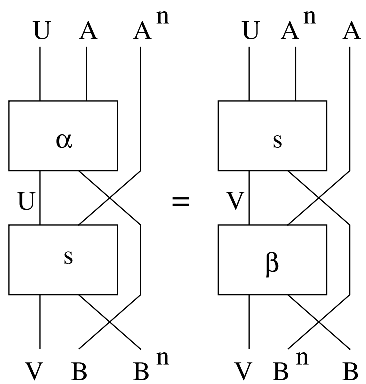
Ignoring run-out states and joining retiming equivalent ones in (U, α) gives rise to a minimal automaton, which is unique up to isomorphism.

Theorem Two finite state Mealy automata are retiming equivalent iff they reduce to the same minimal automaton.

Homomorphism and simulation between automata (U, α) and (V, β) $A \to B$.



Homomorphism



Simulation

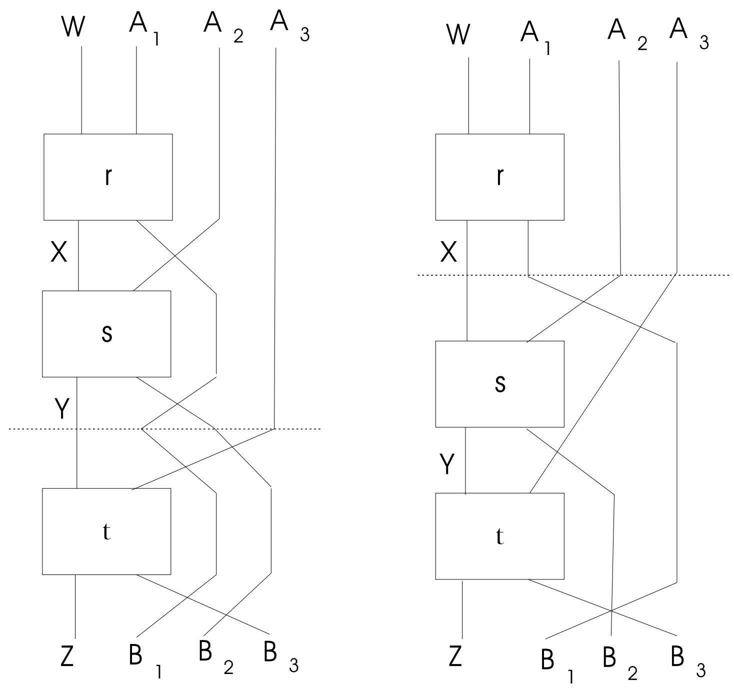
A *simulation* from (U, α) to (V, β) in the category $Circ(\mathcal{M})$ is a morphism $s : U \times A^n \to V \times B^n$ in $\mathcal{M}$, $n \geq 0$, such that

$$cas(\alpha, s) = cas(s, \beta).$$

If $n = 0$, then s is called *immediate*.

Observations:

1. Cascade product of morphisms in $\mathcal{M}$ is associative.



2. Simulations can be composed by the cascade product.

3. There is an identity simulation $1_{(U,\alpha)} = (1_U)_{\mathcal{M}}$ for every automaton (U,α). Consequently, simulations as 2-cells make $Circ(\mathcal{M})$ a 2-category.

4. If s is a simulation from (U,α) to (V,β), then so is $cas(\alpha,s)$. Indeed,

$$cas(\alpha, cas(\alpha, s)) = cas(\alpha, cas(s, \beta)) = cas(cas(\alpha, s), \beta)).$$

Simulations s and $cas(\alpha,s)$ are in fact indistinguishable.

Simulations $s, s' : (U, \alpha) \to (V, \beta)$ are *indistinguishable*, in notation $s \equiv s'$, if there exist integers $k, l \geq 0$ such that

$$cas(\alpha^k, s) = cas(\alpha^l, s').$$

Automata (U, α) and (V, β) are *simulation equivalent* if there exist simulations $s : (U, \alpha) \to (V, \beta)$ and $t : (V, \beta) \to (U, \alpha)$ such that

$$cas(s, t) \equiv 1_{(U,\alpha)} \quad \text{and} \quad cas(t, s) \equiv 1_{(V,\beta)}.$$

Intuitively, the definition says that there exist simulations between automata (U, α) and (V, β) in both directions which are reversible in a certain sense.

Theorem In the category of finite state Mealy automata $Circ(\boldsymbol{Set_f})$, retiming equivalence coincides with simulation equivalence.

The result can be genaralized under some broad conditions regarding the underlying category $\mathcal{M}$, but it does not hold for all monoidal categories.