Two transitive cellular automata and their strictly temporally periodic points

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Outline

1. Background and preliminaries
   - Background
   - Chaos of 1-D CAs in the sense of Devaney

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- The property of global functions, limit behavior, classification, topological structure, computational universality, reversibility and conservation law, etc, are important research issues.

For a one-dimensional (1-D) CA, a point is called strictly temporally periodic (STP) if it is not spatially periodic but temporally periodic. In the following two papers, the set of STP points of 1-D surjective CAs were investigated.

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- the set of STP points is empty if the CA is positively expansive;
- if the CA is additive, the set of STP points can be either dense or empty, and the latter happens iff the CA is topologically transitive;
- the set of STP points has strictly positive measure iff the CA is equicontinuous.

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In this paper, a positive answer for the first problem and a negative answer for the second one are given by showing two classes of topologically mixing reversible 1-D CAs.

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Chaos of 1-D CAs in the sense of Devaney

- A CA $(S^{\mathbb{Z}}, G)$ is called topologically transitive if for any nonempty open subsets U and V of $S^{\mathbb{Z}}$, there exists a nonnegative integer p such that $G^p(U) \cap V \neq \emptyset$, or equivalently by translation invariance and for the reason that cylinders are a basis of the topology,
  for all integers $l \geq 1$, for any words $u, v \in S^l$, there exists a configuration $c \in [u]_0$ and a nonnegative integer p such that $G^p(c) \in [v]_0$.

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- A CA $(S^{\mathbb{Z}}, G)$ is called topologically mixing if for any nonempty open subsets U and V of $S^{\mathbb{Z}}$, there exists a nonnegative integer p such that $G^q(U) \cap V \neq \emptyset$ for all $q \geq p$, or equivalently,
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- Topologically mixing CAs are topologically transitive.

- A CA $(S^{\mathbb{Z}}, G)$ is said to be sensitive to initial conditions if there exist integers $i \leq j$ such that for any configuration c and any integers $k \leq l$, there exist $e \in \mathrm{Cyl}(c, [k, l])$ and a positive integer p such that $G^p(e) \notin \mathrm{Cyl}(G^p(c), [i, j])$.

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- For any CA, (i) implies (iii).
- For reversible CAs, (ii) always holds. So a reversible CA is Devaney-chaotic iff it is topologically transitive.

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Consider first the 1-D CA $A = (\mathbb{Z}, S, N, f_A)$ where S consists of the 5 states shown in Fig. 1, $N = (-1, 0, 1)$, and the local rule is defined as follows: for any configuration $c \in S^{\mathbb{Z}}$, and for any cell $i \in \mathbb{Z}$,

- $F_A(c)(i)$ is in state WALL iff $c(i)$ is in state WALL,

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- $F_A(c)(i)$ has a right arrow if and only if $c(i)$ has a left arrow and $c(i-1)$ is in state WALL, or $c(i)$ is not in state WALL and $c(i-1)$ has a right arrow.

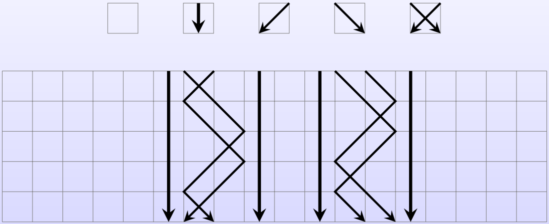

Figure: The states of the CA F_A, where the first state is called BLANK, and the second state is called WALL. The last two states contain a right arrow, and the third and the last states contain a left arrow. The bottom shows a space-time diagram starting at a BLANK-finite configuration.

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P. Kůrka, Languages, equicontinuity and attractors in cellular automata, *Ergodic Theory Dynam. Systems*, 17: 417–433, 1997.

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- In F the stationary walls have become left moving signals, while the right (resp. left) arrows are stationary signals (resp. signals of speed 2 to the left, respectively).
- See Fig. 2 for an example of a space-time diagram.

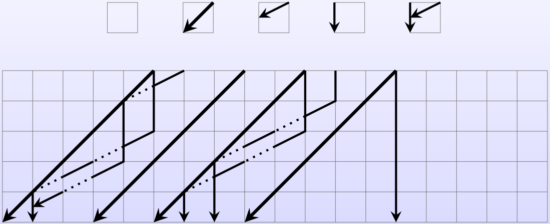

Figure: The top five squares are the states of the CA F. Note that these states are essentially the same as those in Fig. 1, respectively. The bottom shows a space-time diagram of the CA F. The dotted segments are not part of the states, but are included to indicate how the left arrows become signals of speed 2 to the left.

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L. Acerbi, A. Dennunzio, E. Formenti, Conservation of some dynamical properties for operations on cellular automata, *Theoret. Comput. Sci.*, 410: 3685–3693, 2009.

- The set of STP points of CA F is not empty. Indeed, any configuration containing only BLANK states and right arrows (which are stationary signals in F) is a fixed point of F. There are uncountably many choices of such configurations that are not spatially periodic.

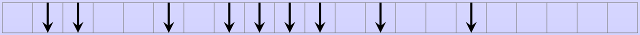


Figure: A fixed point.

- The set of STP points of CA F is not dense: any temporally periodic configuration that contains a cell in state WALL is spatially periodic.

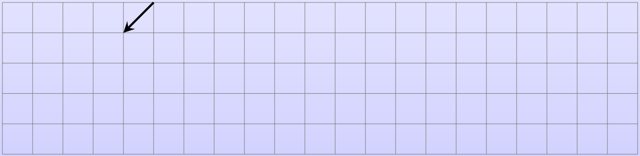
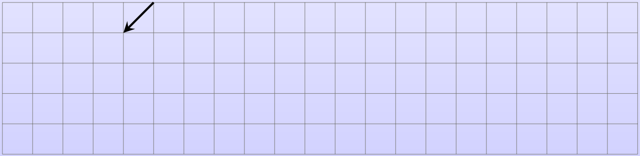


Figure: A STP point.

- The set of STP points of CA F is not dense: any temporally periodic configuration that contains a cell in state WALL is spatially periodic.
- Assume that $F^4(c) = c$ and c has a state WALL.



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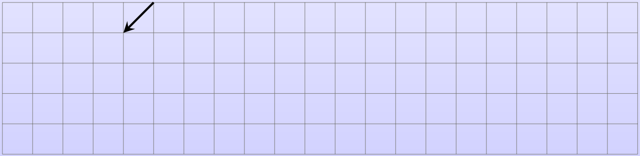

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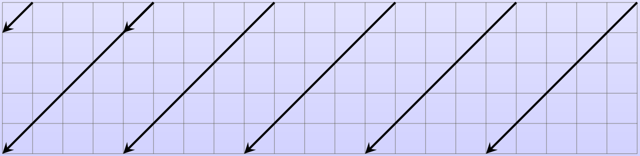

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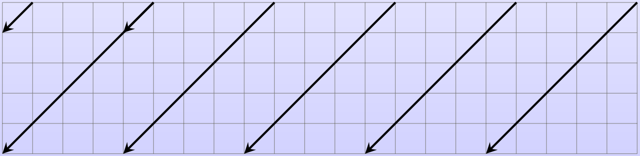

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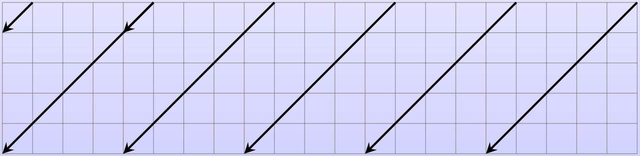

Figure: A STP point.

- Then c is of spatial period $4 \cdot (5^3)!$.

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A topologically mixing reversible 1-D CA whose set of STP points is dense

Consider CA $A = (\mathbb{Z}, S, N, f_A)$ where S consists of the 9 states shown in Fig. 5, $N = (-1, 0, 1)$, and the local rule is defined as follows: in the space-time diagram, each corner is exactly one of the 9 figures shown in Fig. 6. In other words, there are left and right moving signals, colored red and olive, that bounce upon collisions.

Figure: The states of the CA, where the first state is called BLANK. The arrows are colored red and olive.

Figure: The local rule of the CA.

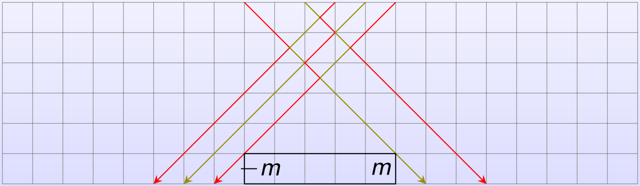
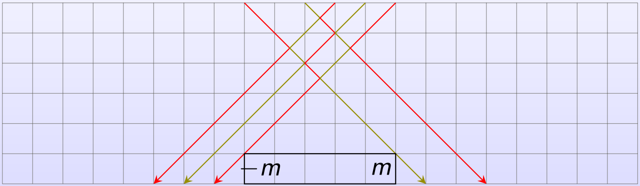


Figure: A "campasses" space-time diagram starting at a BLANK-finite configuration.



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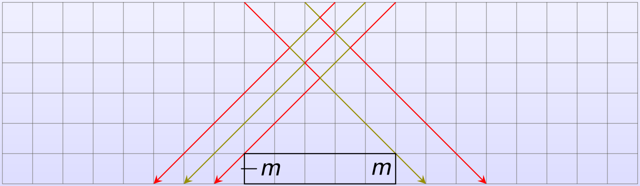

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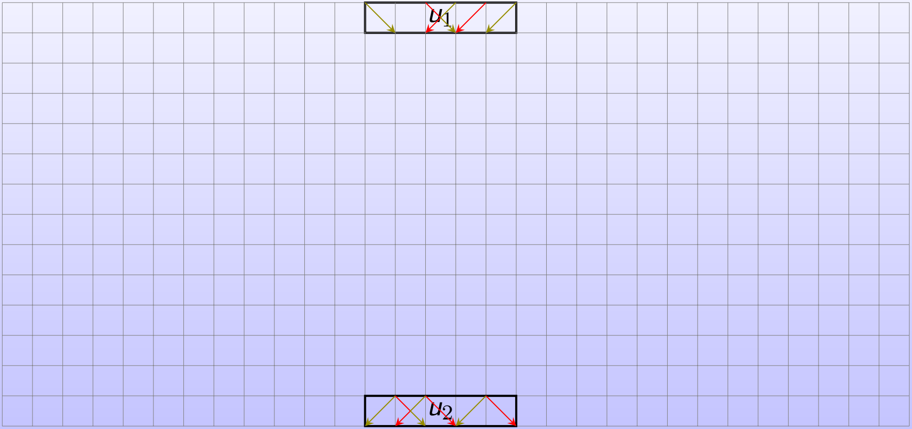
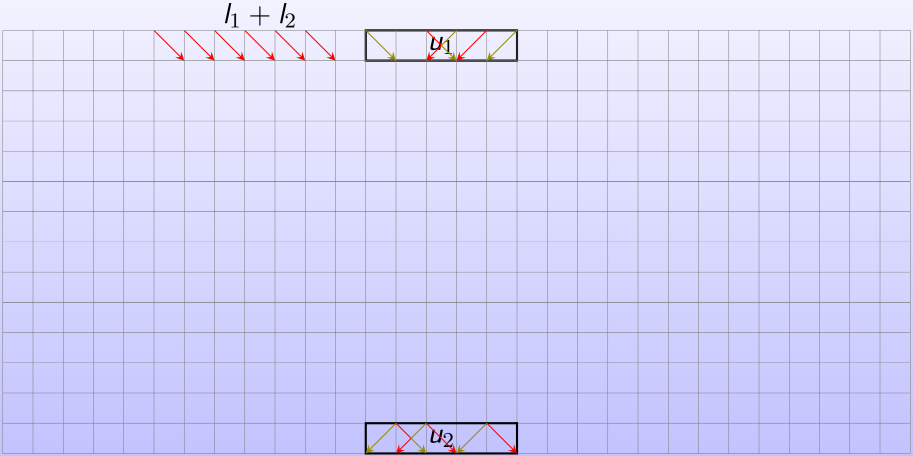
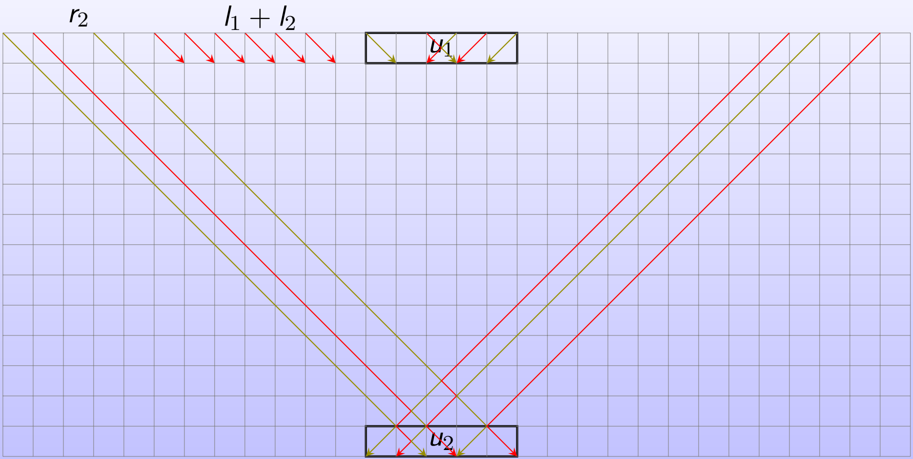
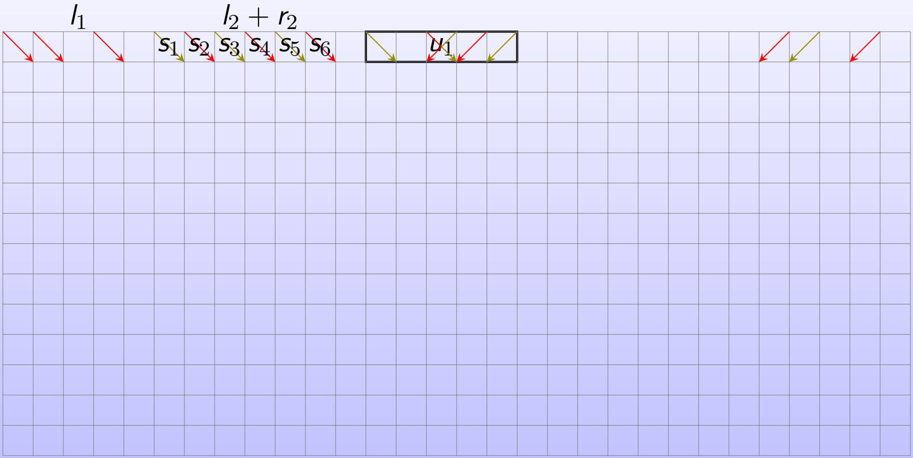

u_1
u_2


$l_1 + l_2$
u_1
u_2


r_2
$l_1 + l_2$
u_1
u_2


l_1
$l_2 + r_2$
s_1 s_2 s_3 s_4 s_5 s_6
u_1

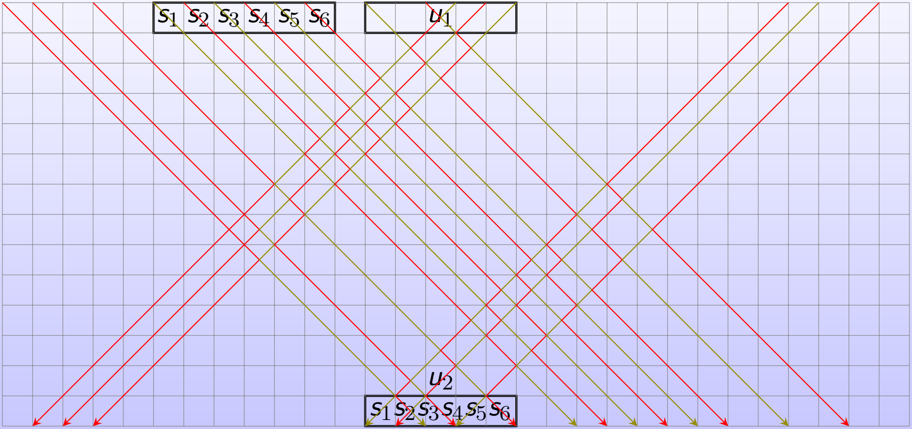


Figure: Transitivity of cylinder $[u_1]_0$ to cylinder $[u_2]_0$

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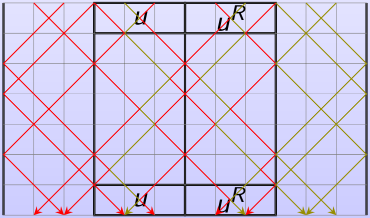

Figure: A subblock of the space-time diagram starting at an STP configuration. The first line denotes the middle part of the initial configuration.

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