Commutators of Bipermutive and Affine Cellular Automata

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Automata 2013

Outline

Motivation

- We study one- and multidimensional *permutive* cellular automata as dynamical systems
- Permutive cellular automata are very chaotic. . .
- . . . and their commutators are sometimes very regular

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- Permutive cellular automata are very chaotic. . .
- . . . and their commutators are sometimes very regular
- [Moore & Boykett 97]: Affine bipermutive CA can only commute with other affine CA
- We generalize this to n dimensions (was left open) using completely different methods
- We also obtain interesting results on orbits of subshifts under permutive CA

Outline

Cellular Automata

Definition

A *cellular automaton* is a function f from $S^{\mathbb{Z}^d}$ to itself defined by a *local rule* $F : S^N \to S$ by

$$f(x)_{\vec{n}} = F(x_{N+\vec{n}}),$$

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Example

The two-dimensional three-neighbor XOR automaton $f : \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$, defined by

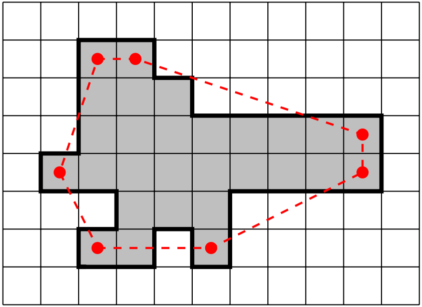
$$f(x)_{\vec{n}} = x_{\vec{n}} + x_{\vec{n}+\vec{e}_1} + x_{\vec{n}+\vec{e}_2} \bmod 2,$$

has neighborhood $\{\vec{0}, \vec{e}_1, \vec{e}_2\}$.

Permutivity

Definition

A cellular automaton f on $S^{\mathbb{Z}^d}$ is *permutive* on a coordinate $\vec{v} \in \mathbb{Z}^d$ if permuting $x_{\vec{v}}$ always permutes $f(x)_{\vec{0}}$. It is *totally extremally permutive* (TEP) if it is permutive in every vertex of the convex hull of its neighborhood. One-dimensional TEP automata are *bipermutive*.



Linearity and Affinity

Definition

Let G be a finite abelian group. A cellular automaton f on $G^{\mathbb{Z}^d}$ is *linear* if $f(x+y) = f(x) + f(y)$, and *affine* if $f(x) = g(x) + c$ for some linear CA g and $c \in G^{\mathbb{Z}^d}$.

Note that c is necessarily unary.

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Example

The three-neighbor two-dimensional XOR automaton is linear. It is also TEP. The XOR automaton composed with a bit flip is affine and TEP.

Outline

Main Lemma

Lemma (Commutator Lemma)

Let f, g and h be cellular automata on $S^{\mathbb{Z}^d}$ such that g and h commute with f. Let $X \subset S^{\mathbb{Z}^d}$ be a subshift such that $\overline{\bigcup_{n \in \mathbb{N}} f^n(X)} = S^{\mathbb{Z}^d}$. If $g|_X = h|_X$, then $g = h$.

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Proof.

Let $y \in S^{\mathbb{Z}^d}$, and let $\epsilon > 0$. Then there exist $n \in \mathbb{N}$ and $x \in X$ such that

$$g(y) \overset{\epsilon}{\approx} g(f^n(x)) = f^n(g(x)) = f^n(h(x)) = h(f^n(x)) \overset{\epsilon}{\approx} h(y),$$

proving the claim. □

Main Lemma

- If $\overline{\bigcup_{n \in \mathbb{N}} f^n(X)} = S^{\mathbb{Z}^d}$ holds, we say f *topologically randomizes* X.
- Related to *asymptotic randomization* of measures (see [Pivato 2012]).

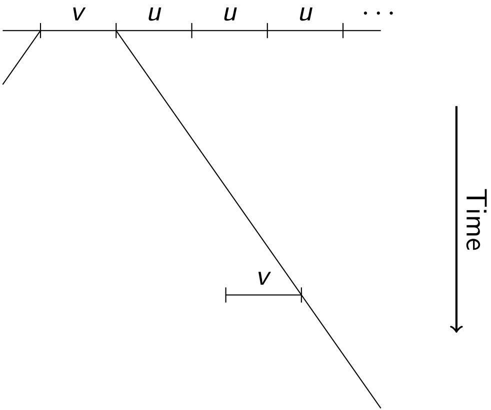
Main Lemma

- If $\overline{\bigcup_{n \in \mathbb{N}} f^n(X)} = S^{\mathbb{Z}^d}$ holds, we say f *topologically randomizes* X.
- Related to *asymptotic randomization* of measures (see [Pivato 2012]).
- It turns out that TEP automata topologically randomize many subshifts.

Outline

Repetition of Patterns

In a bipermutive CA, every word surrounded by a spatially and temporally periodic pattern will repeat:



Randomization of SFTs

Proposition

Let $X \subset S^{\mathbb{Z}}$ be a mixing SFT, let $x \in S^{-\mathbb{N}}$ be such that xs occurs in X for all $s \in S$, and let f be a bipermutive CA on $S^{\mathbb{Z}}$. Then f topologically randomizes X.

We can assume that $x = {}^{\infty}w$ for some $w \in S^{+}$.

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We can assume that $x = {}^{\infty}w$ for some $w \in S^+$.
In the above, a transivite point also exists in X:

Theorem

If a bipermutive CA f topologically randomizes a mixing SFT X, then $\overline{\{f^n(x) \mid n \in \mathbb{N}\}} = S^{\mathbb{Z}}$ for some $x \in X$.

Proof of Proposition

By induction: every $v \in S^*$ occurs in $f^n(X)$ for arbitrarily large n.

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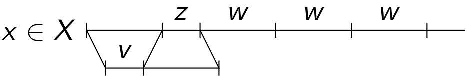
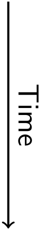


v occurs in nth image of $x \in X$



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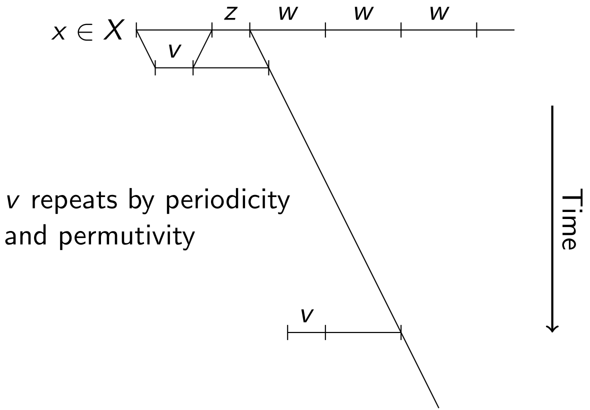

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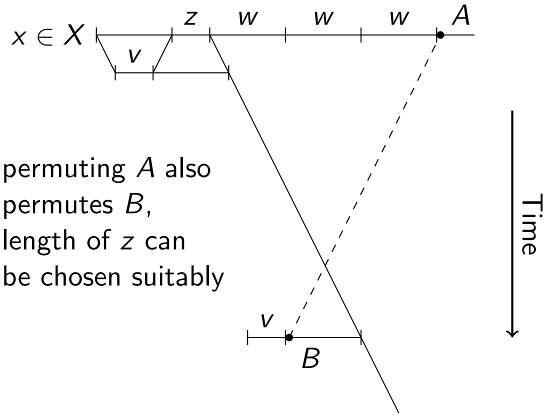

Linear Case and Sparse Shifts

With a similar proof:

Proposition

Every bipermutive linear CA on $\mathbb{Z}_p$, where p is prime, topologically randomizes every nontrivial mixing SFT.

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Corollary

Every bipermutive (bipermutive and linear on $\mathbb{Z}_p$) CA topologically randomizes the one-dimensional k-sparse shift on S (binary k-sparse shift on $\mathbb{Z}_p$).

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Multidimensional Sparse Shifts

Proposition

Every TEP automaton topologically randomizes the d-dimensional k-sparse shift on S (where every k^d-block may contain at most one nonzero symbol).

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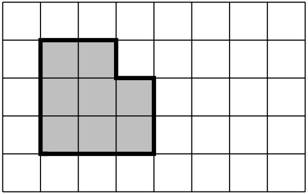
Proof sketch:



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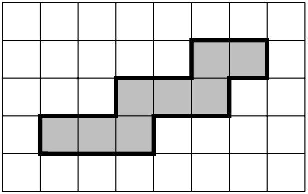

Change neighborhood by applying shear transformation to whole space

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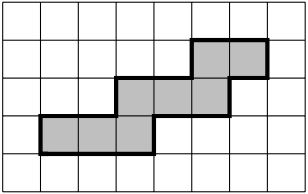

On points with vertical period p, we basically have a bipermutive 1D CA

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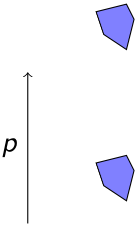

We now take a vertically periodic point in which a desired pattern (repeatedly) appears later...

Multidimensional Sparse Shifts

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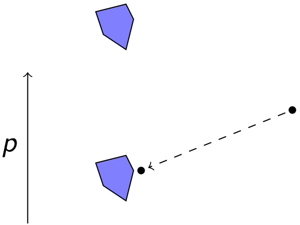


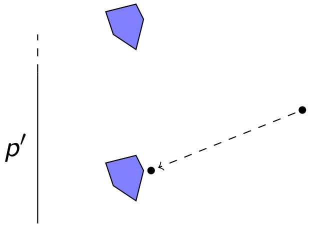
. . . permute one symbol as in the 1D case. . .

Multidimensional Sparse Shifts

Proposition

Every TEP automaton topologically randomizes the d-dimensional k-sparse shift on S (where every k^d-block may contain at most one nonzero symbol).

Proof sketch:



...add an even larger vertical period and continue with the induction

Multidimensional Sparse Shifts

Similarly:

Proposition

Every linear TEP automaton on $\mathbb{Z}_p^{\mathbb{Z}^d}$, where p is prime, topologically randomizes the d-dimensional binary k-sparse shift.

Outline

Size of Commutator

Theorem

Let f be a TEP automaton on $S^{\mathbb{Z}^d}$. The number of CA with given neighborhood of size m that commute with f is at most $|S|^{1+m(|S|-1)}$. If $S = \mathbb{Z}_p$ for a prime p and f is linear, it is at most $|S|^{1+m}$.

Note that there are $|S|^{|S|^m}$ CA with given neighborhood of size m.

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Note that there are $|S|^{|S|^m}$ CA with given neighborhood of size m.

Proof.

Every m-neighbor CA g commuting with f is defined by its restriction to any k-sparse shift X. For large enough k, the local rule of $g|_X$ is defined by $1 + m(|S| - 1)$ patterns. The linear case is similar, but uses binary k-sparse shifts. □

Commutator of Affine TEP CA

Theorem

Let f be an affine TEP automaton on $G^{\mathbb{Z}^d}$, where G is an abelian group, and let g commute with f. Then g is also affine.

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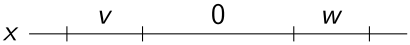
- The 1D case is essentially proved in [Moore & Boykett 97] using algebraic methods.
- We also have a new proof, similar to the randomization of mixing SFTs.
- The multidimensional case is reduced to the one-dimensional case as in the case of randomization.

Our Proof of the 1D Case



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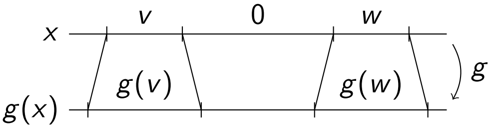

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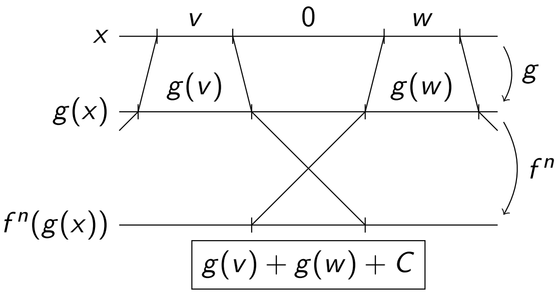

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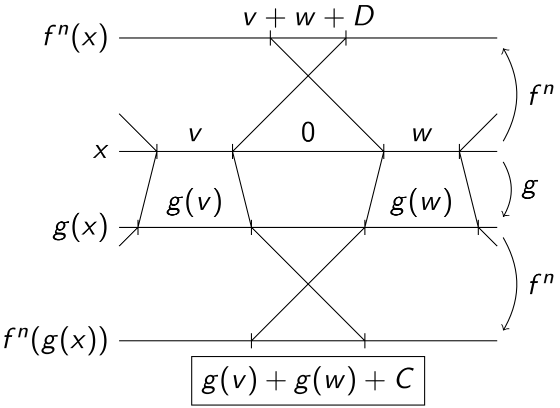

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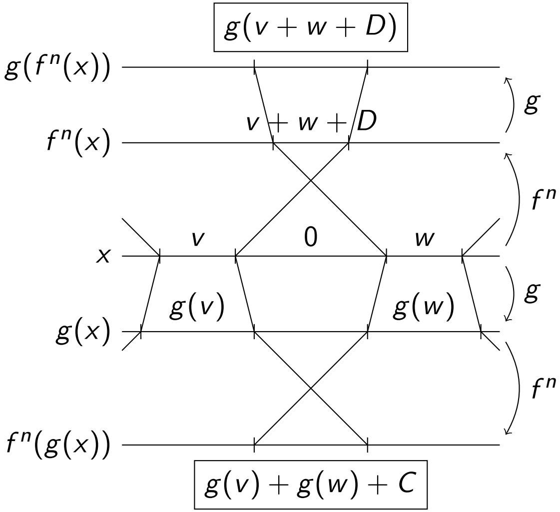

Future Goals

- Show topological randomization for other subshifts and cellular automata
- Study relation to asymptotic randomization of measures
- Generalize final theorem to other algebraic structures

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Thank you!