A guided tour of asynchronous and stochastic cellular automata

Nazim Fates

Inria Nancy, Loria - MalA team
http://www.loria.fr/~fates

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Questions

vocabulary:

στόχος (stochos): goal, aim, target, expectation...
probabilitas: idea of provable, believable, etc. (a probe)

What separates deterministic and stochastic cellular automata?
1st workshop on PCA held in June 2013 (Eindhoven)

- Are probabilistic CA simpler or more complex?
- What can be determined by mathematical analysis? by numerical simulations?
- Are there advantages for using stochastic CAs?
The question of robustness of models

Robustness is a common phenomenon in Nature.

- Where does it come from?
- Can it be transposed to artificial systems?
- If we make a discrete model of a robust phenomenon, is the model robust?

First visual experiments with the *Game of Life*, the *majority rule*, etc.

Is the lack of robustness due to an oversimplification?
(see the PhD thesis of Olivier Bouré)

⇝ how can we make a more systematic study?
Dictyo & the decentralized gathering problem

life cycle of *Dictyostelium discoideum*

spiral waves appear and trigger aggregation

decentralised gathering problem:
particles with limited visibility and no map of the environment have to gather tightly without any centralized control

⇝ discrete stochastic CA-like model ?
A simple model [Automata 2006]

- amoebae attracted by excitation waves (chemotaxis)
- random triggering of waves, firing proba. $\lambda$
- random moves on the amoebae

Open problem: find the optimal value for $\lambda$
separation of two regimes? (single vs. multiple group formation)
Irregular topologies and stochastic rules

models permanent and temporary failures

irregular topology

stochastic

stochastic & irregular
Stochastic Greenberg-Hastings model

Found in chemistry, biology, physiology (heart modelling)...

\[ 0 \quad \rightarrow \quad 1 \quad \rightarrow \quad \ldots \quad \rightarrow \quad M-1 \quad \rightarrow \quad M \]

excitation received

control parameter: probability of transmission \( p_T \)

extinct and active regimes; sharp separation; ex. TV4, \( M=4 \)

A) B)

\( p_T = \)

\( p_T = \)

Simulation time \( t \)
Is this phenomenon robust? : excitation level

- directed percolation is conserved
- critical threshold increases moderately

**open problem:**
even-odd same threshold, for von Neumann neighb. only
What is the limit for $M \rightarrow \infty$?
Is this phenomenon robust? : changing the topology

- VN : von Neumann
- Moo : Moore
- HX : hexagonal
- Cl : circular

Critical value $p_T$ increases as more links or sites are removed always DP but difficulty to locate the phase transition

Mixing of classical percolation and directed percolation?
mean-field analysis seems to confirm inverse proportionality law

\[ p_{TC} \sim 2.05 \times N^{-1.05} \]

What about binary systems?
From fully asynchronous to $\alpha$-asynchronous updating...

partial extension to $\alpha$-asynchronous updating was done...
some rules with brutal changes of behaviours pose problem

$n = 10\,000, T = 10\,000$

What happens for $\alpha \sim 0.62$?

magnetisation vs. temp.

$M \sim (T_c - T)^\gamma$

2nd order phase transitions

critical exponents

universality classes
Rule BCDEFGH - 50: \( f(a, b, c) = \begin{cases} \bar{b} & \text{if } (a, b, c) \neq (0, 0, 0) \\ 0 & \text{otherwise} \end{cases} \)

- **healthy** cell (0) infected by infected neighbour
- **infected** cell (1) always recovers

\[ \alpha = 0.60 \]
Directed percolation

From [Hinrichsen, 2000]

isotropic: $p_c = 1/2$, directed: $p_c < 1/2$.

For a random initial configuration: $d \sim t^{-\delta}$ with $\delta = 0.1595$
Results near criticality

\[ n = 20,000 \text{ averages made on 100 runs} \]

ECA 50: Log-Log plot of \( d(t) \) for different values of \( \alpha \)

\[ f(t) = K \cdot t^{-0.1595} \]

\( \alpha = 0.626 \)
\( \alpha = 0.627 \)
\( \alpha = 0.628 \)
\( \alpha = 0.629 \)
\( \alpha = 0.630 \)

confirms directed percolation hypothesis (for nine ECA)
Analytical techniques

Restrictions to **double-quiescent** ECA under **fully asynchronous** updating
Note that fixed points under synchronous and asynchronous updating are identical, but not cycles!

Questions:

- What is the set of reachable fixed points?
- What is the probability to reach a given fixed point?
- How to link behaviour and convergence time?
A classification according to the convergence time

How to quantify the convergence time of a fully asynchronous CA?

▶ we rescale time by a factor $1/n$
▶ $T_x$ is the convergence time starting from $x \in Q^L$
▶ we calculate $E[T_x]$ for all updatings
▶ we extract the maximum over all configurations of size $n$

we obtain... the worst expected convergence time in average:

$$WECT = \max_{x \in Q^L} E[T_x]$$

Theorem

Among the 25 double-quiescent ECA, 21 converge a.s., 4 diverge.

$WECT$ is:

$$0, \ln n, \Theta\{n\}, \Theta\{n^2\}, \Theta\{2^n\}, \infty$$
### A classification based on the WECT

<table>
<thead>
<tr>
<th>behaviour</th>
<th>ACE (#)</th>
<th>rule</th>
<th>01</th>
<th>10</th>
<th>010</th>
<th>101</th>
<th>convergence</th>
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<tr>
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</tbody>
</table>
Explanations: **Linear** WECT

- the regions of 1s can only grow or only shrink,
- one region performs RW, the other only grows or shrinks, fusion of regions are allowed.

<table>
<thead>
<tr>
<th>Tcode</th>
<th>W</th>
<th>s</th>
<th>01</th>
<th>10</th>
<th>E</th>
<th>D</th>
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<td>B</td>
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<td>+</td>
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<tr>
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<td>EFG</td>
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<td>BEFG</td>
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<td>→</td>
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<td>D</td>
</tr>
</tbody>
</table>
Explanations: **Quadratic** WECT

- One or two regions perform a random walk, fusions allowed.

<table>
<thead>
<tr>
<th>Tcode</th>
<th>W</th>
<th>s</th>
<th>01</th>
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<tbody>
<tr>
<td>BDEF</td>
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<td>·</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>BDEG</td>
<td>170</td>
<td>2</td>
<td>←</td>
<td>←</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>BCDEFG</td>
<td>178</td>
<td>1</td>
<td>⇓</td>
<td>⇓</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

- Same but, only **one fusion** is allowed.

<table>
<thead>
<tr>
<th>Tcode</th>
<th>W</th>
<th>s</th>
<th>01</th>
<th>10</th>
<th>E</th>
<th>D</th>
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<tbody>
<tr>
<td>BEF</td>
<td>194</td>
<td>4</td>
<td>⇓</td>
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<td>+</td>
<td>·</td>
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<tr>
<td>BEG</td>
<td>170</td>
<td>2</td>
<td>←</td>
<td>←</td>
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<tr>
<td>BCEFG</td>
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<td>⇓</td>
<td>⇓</td>
<td>+</td>
<td>·</td>
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</tbody>
</table>

No direct martingales here, “bouncing” phenomena
Explanations: non-converging, recurrent rules

- no fusion is allowed, one or two regions perform a random walk

<table>
<thead>
<tr>
<th>Tcode</th>
<th>W</th>
<th>s</th>
<th>01</th>
<th>10</th>
<th>E</th>
<th>D</th>
</tr>
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<tbody>
<tr>
<td>BF</td>
<td>198</td>
<td>2</td>
<td>⇐⇑⇑</td>
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<tr>
<td>BG</td>
<td>142</td>
<td>2</td>
<td>⇐⇑⇑</td>
<td>←</td>
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</tr>
<tr>
<td>BCF</td>
<td>214</td>
<td>2</td>
<td>⇐⇑⇑</td>
<td>→</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>BCFG</td>
<td>150</td>
<td>1</td>
<td>⇐⇑⇑</td>
<td>⇐⇑⇑</td>
<td>.</td>
<td>.</td>
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</tbody>
</table>

⇝ Is that all?
Rule BCEF

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
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<tbody>
<tr>
<td>000</td>
<td>001</td>
<td>100</td>
<td>101</td>
<td>010</td>
<td>011</td>
<td>110</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

01 frontiers perform a random walk, the 10 frontiers always extend to the right, only 1-regions can disappear

![Diagram](image)

proba to increase by 1 is $2\epsilon$, proba to decrease by 1 is $\epsilon$. biased random walk in the “wrong direction”

Relaxation time $\tau$ scales as $2^n$: metastability
Beyond double-quiescent ECA

**many open questions**, guesses based on observations:

Logarithmic convergence:

- to a homogeneous fixed point: 16 min. rules
  - BCEGH - 26
  - BCDEGH - 58
  - BEH - 74

- to a heterogeneous fixed point: 10 min. rules
  - AFGH - 5
  - EH - 72
  - DEH - 104

~~ How to unify the analysis ? (new lemma ?)
Polynomial convergence

- **Linear convergence**: only 2 rules!
  - BH - 78
  - BCH - 94
  - shows the necessity to start from a zone of 1s

- **Quadratic convergence**: 10 rules, production of stripes
  - BFGH - 6
  - ADFGH - 37
  - ADGH - 45
  - strange case of ECA BDFGH - 38
Non-converging rules

- no fixed point (13 rules)

<table>
<thead>
<tr>
<th>Code</th>
<th>Rule 1</th>
<th>Rule 2</th>
<th>Rule 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>AEFGH</td>
<td>(+ A, + H)</td>
<td>(+ D \text{ or } E)</td>
<td>(+ B \text{ or } E \text{ or } C)</td>
</tr>
<tr>
<td>ABDEGH</td>
<td>(+ A, + H)</td>
<td>(+ D \text{ or } E)</td>
<td>(+ G \text{ or } D \text{ or } F)</td>
</tr>
<tr>
<td>ACDEGH</td>
<td>(+ B \text{ or } F \text{ or } G \text{ or } C)</td>
<td>(+ B \text{ or } F \text{ or } G \text{ or } C)</td>
<td></td>
</tr>
</tbody>
</table>
Metastable or non-converging rules?

- partial stabilization with localised evolution (4 rules)
  - CGH - 28
  - AEH - 73
  - DH - 108

- noise-like evolution (9 rules)
  - BDGH - 46
  - CDGH - 60
  - BDH - 110

0 only fixed point for all rules but one (ADEH)
**but**, for 6 rules, 0 non-reachable: E absent from T-code
for ADEH - 105, one non-reachable fixed point \((1001)^n/4\)

\(\rightsquigarrow\) How to show exponential convergence of 90 (XOR2), 122?
The two-dimensional case

much more difficult to study analytically...first attempts:

► with Gerin on totalistic vN neighb. (Automata 2008)
► Regnault et al., minority rule

example of open problem:

► contamination rule: show that the WECT scales as $L = \sqrt{n}$

various behaviours: see the paper in the proceedings for an empirical classification that refines [Fatès & Gerin, JCA 2009]

⇝ How to distinguish between exponential, and infinite WECT?

planning techniques were used to prove “automatically” the metastability of totalistic rule 10 [Höffmann et. al., 2010]
Synthesis

asynch. CAs in between deterministic and “pure” stochastic CAs

**major open problem**:
find analytical techniques to study the metastable rules

next steps to dive into the stochastic CA universe:
- stochastic “blend” of two rules (8808 couples)
- full exploration of the eight-dimensional hypercube
- tackle inverse problems

⇝ a creative role of randomness?
Simon Hantai (blue)
Simon Hantai (red)
Simon Hantai (yellow)
Simon Hantai (multicolor)