A guided tour of asynchronous and stochastic cellular automata

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Questions

vocabulary :
στόχος (stochos): goal, aim, target, expectation...
probabilitas : idea of provable, believable, etc. (a probe)

What separates deterministic and stochastic cellular automata ?
1st workshop on PCA held in June 2013 (Eindhoven)

- Are probabilistic CA simpler or more complex ?
- What can be determined by mathematical analysis ? by numerical simulations ?
- Are there advantages for using stochastic CAs ?

The question of robustness of models

Robustness is a common phenomenon in Nature.

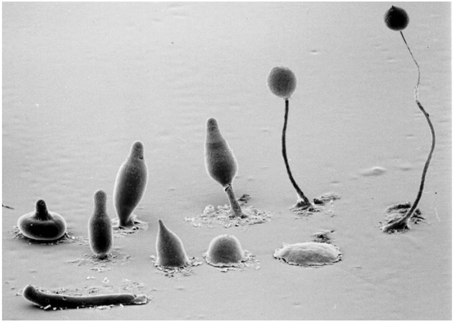
- Where does it come from ?
- Can it be transposed to artificial systems ?
- If we make a discrete model of a robust phenomenon, is the model robust ?

First visual experiments with the *Game of Life*, the *majority rule*, etc.

Is the lack of robustness due to an oversimplication ?
(see the PhD thesis of Olivier Bouré)

$\rightsquigarrow$ how can we make a more systematic study ?

Dictyo & the decentralized gathering problem



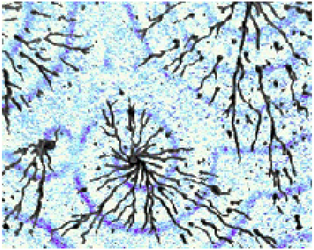
(c) Mark Grimson et Larry Blanton.

life cycle of *Dictyostelium discoideum*
spiral waves appear and trigger aggregation

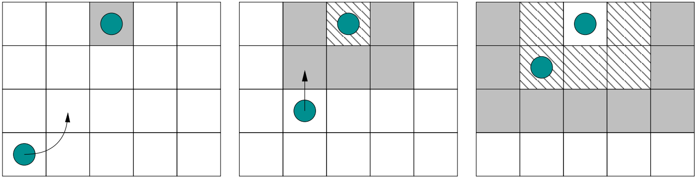
decentralised gathering problem:
particles with limited visibility and no map of the environment have to gather tightly without any centralized control

$\rightsquigarrow$ discrete stochastic CA-like model ?

A simple model [Automata 2006]



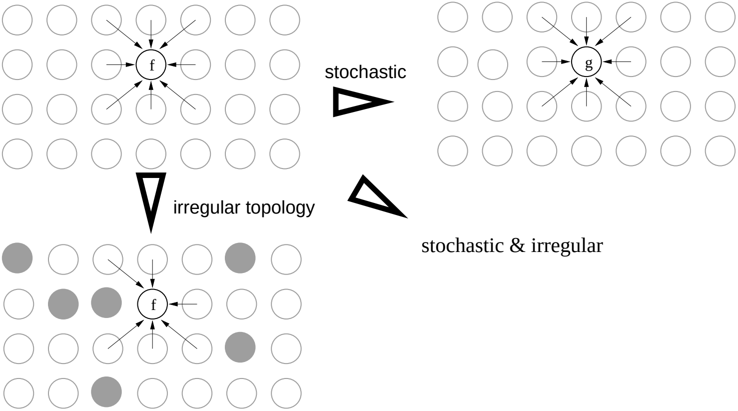
- amoebae attracted by excitation waves (chemotaxis)
- random triggering of waves, firing proba. λ
- random moves on the amoebae



open problem : find the optimal value for λ
separation of two regimes ? (single vs. multiple group formation)

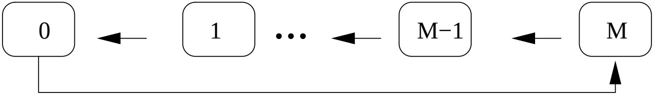
Irregular topologies and stochastic rules

models permanent and temporary failures



Stochastic Greenberg-Hastings model

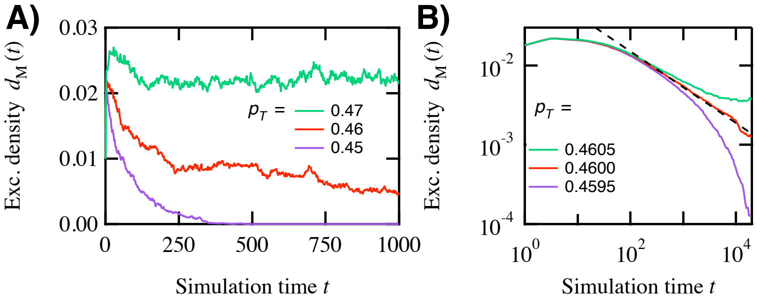
Found in chemistry, biology, physiology (heart modelling)...



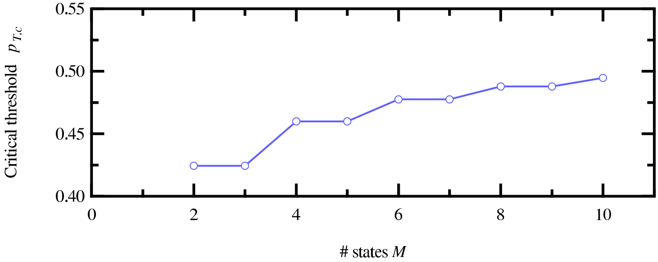
excitation received

control parameter : probability of transmission p_T

extinct and **active** regimes ; sharp separation ; ex. TV4, M=4



Is this phenomenon robust ? : excitation level



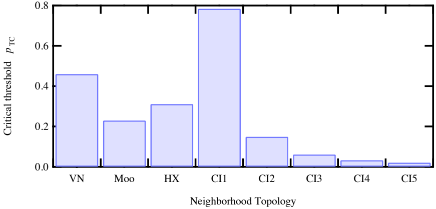
- directed percolation is conserved
- critical threshold increases moderately

open problem:

even-odd same threshold, for von Neumann neighb. only

What is the limit for $M \to \infty$?

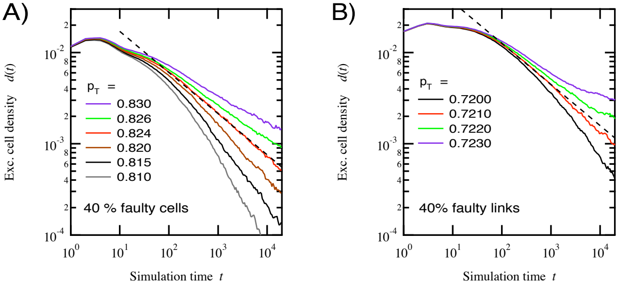
Is this phenomenon robust ? : changing the topology



- VN : von Neumann
- Moo : Moore
- HX : hexagonal
- CI : circular

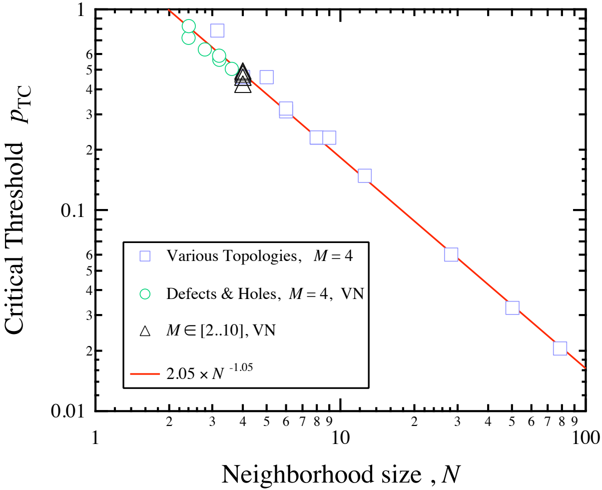
critical value p_T increases as more links or sites are removed

always DP **but** difficulty to locate the phase transition



mixing of classical percolation and directed percolation ?

Analysis does help

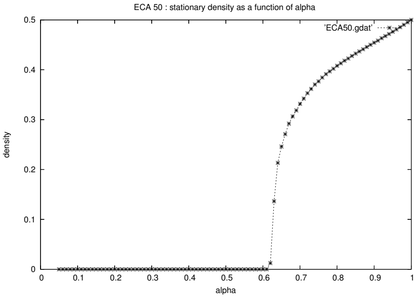


mean-field analysis seems to confirm inverse proportionality law

$\rightsquigarrow$ What about binary systems ?

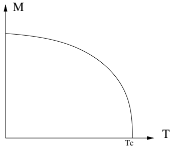
From fully asynchronous to α-asynchronous updating...

partial extension to α-asynchronous updating was done...
some rules with brutal changes of behaviours pose problem



$n = 10\,000$, $T = 10\,000$
What happens for $\alpha \sim 0.62$?

magnetisation vs. temp.
$M \sim (T_c - T)^\gamma$

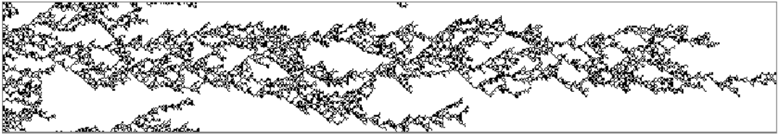


2nd order phase transitions
critical exponents
universality classes

Close-up on ECA 50

Rule BCDEFGH - 50: $f(a,b,c) = \begin{cases} \bar{b} & \text{if } (a,b,c) \neq (0,0,0) \\ 0 & \text{otherwise} \end{cases}$

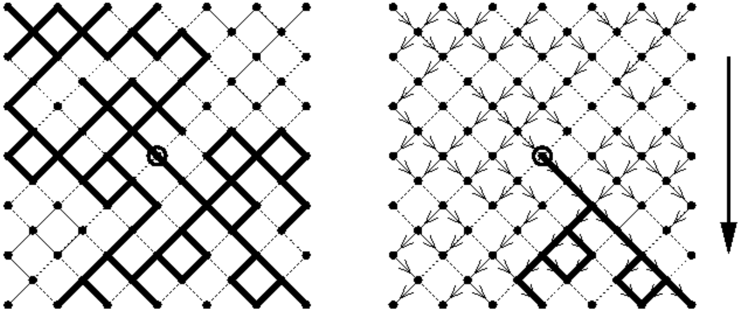
- **healthy** cell (0) infected by infected neighbour
- **infected** cell (1) always recovers



$\alpha = 0.60$

Directed percolation

From [Hinrichsen,2000]

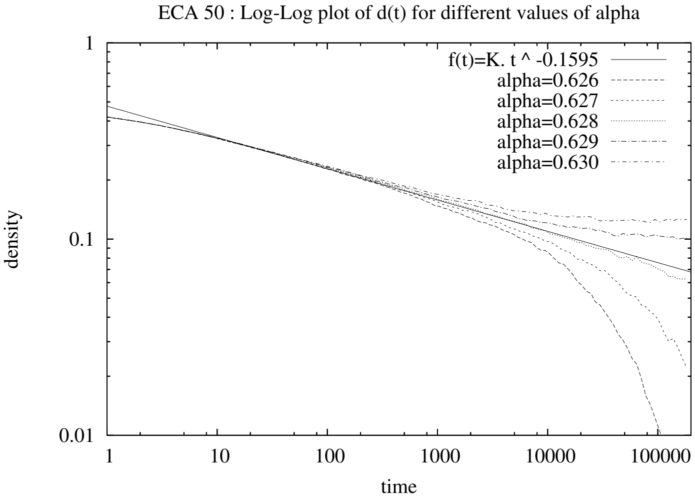


isotropic bond percolation *directed bond percolation*

isotropic: $p_c = 1/2$, directed: $p_c < 1/2$.
For a random intial configuration: $d \sim t^{-\delta}$ with $\delta = 0.1595$

Results near criticality

$n = 20\,000$ averages made on 100 runs



confirms directed percolation hypothesis (for nine ECA)

Analytical techniques

Restrictions to **double-quiescent** ECA under **fully asynchronous** updating
Note that fixed points under synchronous and asynchronous updating are identical, but not cycles !

Questions:

- What is the set of reachable fixed points?
- What is the probability to reach a given fixed point?
- How to link behaviour and convergence time?

A classification according to the convergence time

How to quantify the convergence time of a fully asynchronous CA ?

- we rescale time by a factor $1/n$
- T_x is the convergence time starting from $x \in Q^{\mathcal{L}}$
- we calculate $E[T_x]$ for all updatings
- we extract the maximum over all configurations of size n

we obtain... **the worst expected convergence time in average**:

$$WECT = max_{\{x \in Q^{\mathcal{L}}\}} E[T_x]$$

Theorem

Among the 25 double-quiescent ECA, 21 converge a.s., 4 diverge. WECT is :

$$0, \ln n, \Theta\{n\}, \Theta\{n^2\}, \Theta\{2^n\}, \infty$$

A classification based on the WECT

behaviour	ACE (#)	rule	01	10	010	101	convergence
identity	204 (1)	∅	·	·	·	·	0
coupon collector	200 (2)	E	·	·	+	·	$\Theta(\ln n)$
	232 (1)	DE	·	·	+	+	
"monotone"	206 (4)	B	$\leftarrow$	·	·	·	$\Theta(n)$
	132 (2)	BC	$\leftarrow$	$\rightarrow$	·	·	
	234 (4)	BDE	$\leftarrow$	·	+	+	
	250 (2)	BCDE	$\leftarrow$	$\rightarrow$	+	+	
	202 (4)	BE	$\leftarrow$	·	+	·	
	192 (4)	EF	$\rightarrow$	·	+	·	
	218 (2)	BCE	$\leftarrow$	$\rightarrow$	+	·	
	128 (2)	EFG	$\rightarrow$	$\leftarrow$	+	·	
biased random walk	242 (4)	BCDEF	$\leftrightsquigarrow$	$\rightarrow$	+	+	
	130 (4)	BEFG	$\leftrightsquigarrow$	$\leftarrow$	+	·	
random walk	226 (2)	BDEF	$\leftrightsquigarrow$	·	+	+	$\Theta(n^2)$
	170 (2)	BDEG	$\leftarrow$	$\leftarrow$	+	+	
	178 (1)	BCDEFG	$\leftrightsquigarrow$	$\leftrightsquigarrow$	+	+	
	194 (4)	BEF	$\leftrightsquigarrow$	·	+	·	
	138 (4)	BEG	$\leftarrow$	$\leftarrow$	+	·	
	146 (2)	BCEFG	$\leftrightsquigarrow$	$\leftrightsquigarrow$	+	·	
biased random walk	210 (4)	BCEF	$\leftrightsquigarrow$	$\rightarrow$	+	·	$\Theta(2^n)$
no fixed-point convergence	198 (2)	BF	$\leftrightsquigarrow$	·	·	·	other type
	142 (2)	BG	$\leftarrow$	$\leftarrow$	·	·	
	214 (4)	BCF	$\leftrightsquigarrow$	$\rightarrow$	·	·	
	150 (1)	BCFG	$\leftrightsquigarrow$	$\leftrightsquigarrow$	·	·	

Explanations: **Linear** WECT

- the regions of 1s can only grow or only shrink,
- one region performs RW, the other only grows or shrinks, fusion of regions are allowed.

Tcode	W	s	01	10	E	D
B	206	4	$\leftarrow$	$\cdot$	$\cdot$	$\cdot$
BC	132	2	$\leftarrow$	$\rightarrow$	$\cdot$	$\cdot$
BDE	234	4	$\leftarrow$	$\cdot$	$+$	$+$
BCDE	250	2	$\leftarrow$	$\rightarrow$	$+$	$+$
BE	202	4	$\leftarrow$	$\cdot$	$+$	$\cdot$
EF	192	4	$\rightarrow$	$\cdot$	$+$	$\cdot$
BCE	218	2	$\leftarrow$	$\rightarrow$	$+$	$\cdot$
EFG	128	2	$\rightarrow$	$\leftarrow$	$+$	$\cdot$
BCDEF	242	4	$\leftrightsquigarrow$	$\rightarrow$	$+$	$+$
BEFG	130	4	$\leftrightsquigarrow$	$\leftarrow$	$+$	$\cdot$

Explanations: **Quadratic** WECT

- One or two regions perform a radom walk, fusions allowed.

Tcode	W	s	01	10	E	D
BDEF	226	2	$\leftrightsquigarrow$	$\cdot$	$+$	$+$
BDEG	170	2	$\leftarrow$	$\leftarrow$	$+$	$+$
BCDEFG	178	1	$\leftrightsquigarrow$	$\leftrightsquigarrow$	$+$	$+$

- Same but, only **one fusion** is allowed.

Tcode	W	s	01	10	E	D
BEF	194	4	$\leftrightsquigarrow$	$\cdot$	$+$	$\cdot$
BEG	170	2	$\leftarrow$	$\leftarrow$	$+$	$\cdot$
BCEFG	178	1	$\leftrightsquigarrow$	$\leftrightsquigarrow$	$+$	$\cdot$

No direct martingales here, “bouncing” phenomena

Explanations : non-converging, recurrent rules

- no fusion is allowed, one or two regions perform a random walk

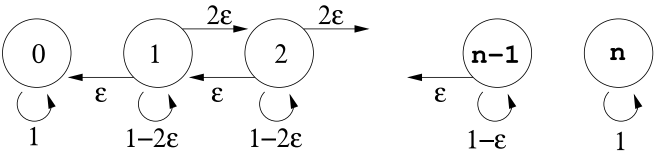
Tcode	W	s	01	10	E	D
BF	198	2	$\leftrightsquigarrow$	$\cdot$	$\cdot$	$\cdot$
BG	142	2	$\leftrightsquigarrow$	$\leftarrow$	$\cdot$	$\cdot$
BCF	214	2	$\leftrightsquigarrow$	$\rightarrow$	$\cdot$	$\cdot$
BCFG	150	1	$\leftrightsquigarrow$	$\leftrightsquigarrow$	$\cdot$	$\cdot$

$\rightsquigarrow$ Is that all ?

Rule BCEF

A	B	C	D	E	F	G	H
000	001	100	101	010	011	110	111
0	1	1	0	0	0	1	1

01 frontiers perform a random walk, the 10 frontiers always extend to the right, only 1-regions can disappear



proba to increase by 1 is 2ϵ, proba to decrease by 1 is ϵ.
biased random walk in the "wrong direction"

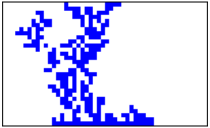
Relaxation time τ scales as 2^n : **metastability**

Beyond double-quiescent ECA

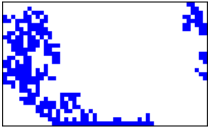
many open questions, guesses based on observations:

Logarithmic convergence:

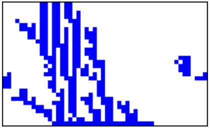
- to a homogeneous fixed point: 16 min. rules



BCEGH - 26

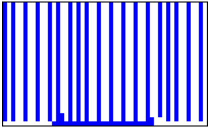


BCDEGH - 58

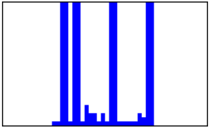


BEH - 74

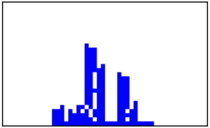
- to a heterogeneous fixed point: 10 min. rules



AFGH - 5



EH - 72



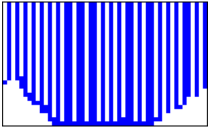
DEH - 104

$\rightsquigarrow$ How to unify the analysis ? (new lemma ?)

Polynomial convergence

- Linear convergence : only 2 rules !

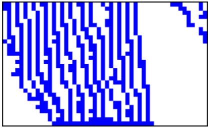
BH - 78



BCH - 94

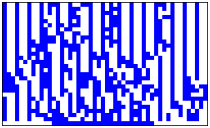
$\rightsquigarrow$ shows the necessity to start from a zone of 1s

- Quadratic convergence : 10 rules, production of stripes



BFGH - 6

ADFGH - 37

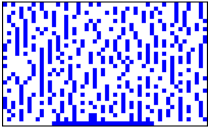


ADGH - 45

$\rightsquigarrow$ strange case of ECA BDFGH - 38

Non-converging rules

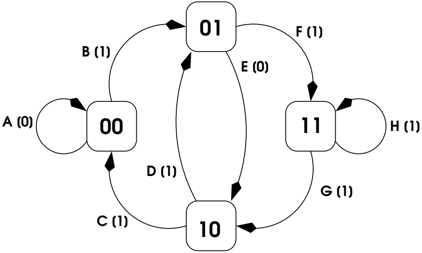
- no fixed point (13 rules)



AEFGH - 1

ABDEGH - 43

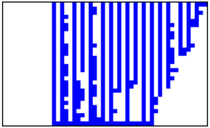
ACDEGH - 57



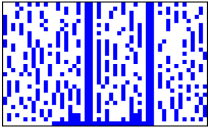
T-code:
+ A, + H
+ D or E
+ B or E or C
+ G or D or F
+ B or F or G or C

Metastable or non-converging rules ?

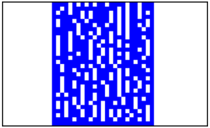
- partial stabilization with localised evolution (4 rules)



CGH - 28

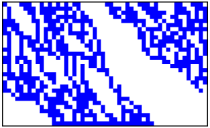


AEH - 73



DH - 108

- noise-like evolution (9 rules)



BDGH - 46

CDGH - 60

BDH - 110

0 only fixed point for all rules but one (ADEH)
but, for 6 rules, **0** non-reachable: E absent from T-code
for ADEH - 105, one non-reachable fixed point $(1001)^{n/4}$

$\rightsquigarrow$ How to show exponential convergence of 90 (XOR2), 122 ?

The two-dimensional case

much more difficult to study analytically...first attempts:

- with Gerin on totalistic vN neighb. (Automata 2008)
- Regnault et al., minority rule

example of open problem:

- contamination rule : show that the WECT scales as $L = \sqrt{n}$

various behaviours : see the paper in the proceedings for an empirical classification that refines [Fatès & Gerin, JCA 2009]

$\rightsquigarrow$ How to distinguish between **exponential**, and **infinite** WECT ?

planning techniques were used to prove “automatically” the metastability of totalistic rule **10** [Höffmann et. al., 2010]

Synthesis

asynch. CAs in between deterministic and "pure" stochastic CAs

major open problem :
find analytical techniques to study the metastable rules

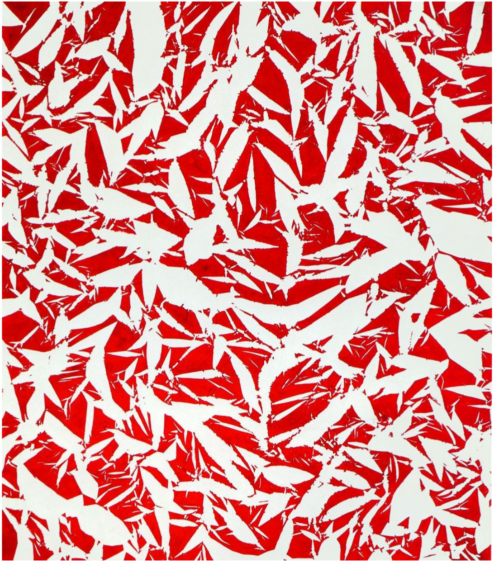
next steps to dive into the stochastic CA universe:

- stochastic "blend" of two rules (8808 couples)
- full exploration of the eight-dimensional hypercube
- tackle inverse problems

$\rightsquigarrow$ a creative role of randomness ?

Simon Hantai (blue)

Simon Hantai (red)



Simon Hantai (yellow)

Simon Hantai (multicolor)