Turing, symmetry breakings and patterns

A tutorial on stochastic cellular automata

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Turing 1952

Philosophical Transactions of the Royal Society of London

Series B, Biological Sciences, Vol. 237, 1952, pp. 37-72.

THE CHEMICAL BASIS OF MORPHOGENESIS

By A. M. TURING, F.R.S. *University of Manchester*

(*Received* 9 *November* 1951—*Revised* 15 *March* 1952)

It is suggested that a system of chemical substances, called morphogens, reacting together and diffusing through a tissue, is adequate to account for the main phenomena of morphogenesis. Such a system, although it may originally be quite homogeneous, may later develop a pattern or structure due to an instability of the homogeneous equilibrium, which is triggered off by random disturbances. Such reaction-diffusion systems are considered in some detail in the case of an isolated ring of cells, a mathematically convenient, though biologically unusual system. The investigation is chiefly concerned with the onset of instability. It is found that there are six essentially different forms which this may take. In the most interesting form stationary waves appear on the ring. It is suggested that this might account, for instance, for the tentacle patterns on *Hydra* and for whorled leaves. A system of reactions and diffusion on a sphere is also considered. Such a system appears to account for gastrulation. Another reaction system in two dimensions gives rise to patterns reminiscent of dappling. It is also suggested that stationary waves in two dimensions could account for the phenomena of phyllotaxis.

The purpose of this paper is to discuss a possible mechanism by which the genes of a zygote may determine the anatomical structure of the resulting organism. The theory does not make any new hypotheses; it merely suggests that certain well-known physical laws are sufficient to account for many of the facts. The full understanding of the paper requires a good knowledge of mathe-

Morphogenesis in Nature

src: www.images-photos-plongee.com

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" Such a system, although it may originally be quite homogeneous, may later develop a pattern or structure due to an instability of the homogeneous equilibrium, which is triggered off by random disturbances."

Turing, 1952

Spatial vs. temporal symmetry breaking

sand riddles

vs.

pendulum synchronisation:
Huygens (1629-1695),
Leibniz (1646-1716)

src: www.kazeo.com

src: www.nd-de-graces.com

Three Turing symmetry breaking games

The state game :
Design a rule that acts similarly on 0 and 1 such that:
the system (almost always) reaches an all-0 or all-1 fixed point

The pattern game:
Design a rule that such that:
the system (almost always) reaches 010101... or a checkerboard

The synchronization game:
Design a rule that such that:
the system (almost always) reaches a $\mathbf{0} \leftrightarrow \mathbf{1}$ cycle

Tutorial map

State game in 1D, asynch. rules:

- rule space and notations
- majority
- shift
- FNE & Traffic

State game in 1D, PCA rules:

- stochastic blend rules

State game in 2D:

- Toom's Rule

Pattern game:

- "Buren" rules

Synchronisation game:

- synchronising rules

Majority rule : coupon collector

Rule : Take the state that is most present in your neighbourhood

A	B	C	D	E	F	G	H
000	001	100	101	010	011	110	111
0	0	0	1	0	1	1	1

- rule E

Majority rule : coupon collector

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- rule E
  1. remove isolated 1s (010)
  2. for k isolated ones (010) , prob. to remove is $\frac{k}{n}$
  3. average *steps* before convergence:

$$T_k(n) = \frac{n}{k} + \frac{n}{k-1} + \ldots + \frac{n}{1} \sim n \ln n$$

- Rule DE : similar with possible decrease by 3.

t	0	1	$\underline{0}$	1	0
$t+1$	0	1	1	1	0

$\rightsquigarrow$ Fast (too fast ?) convergence to fixed point : other neighb. ?

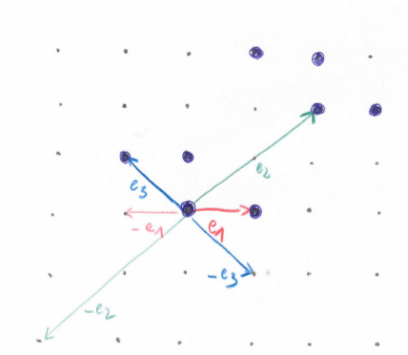
Back

The "symmetric" majority rule in any dimension

Lemma (Marcovici et al., EJP, 2013)

Consider the majority rule on $\mathbb{Z}^d$ with a symmetric neighbourhood that includes the cell itself: $\mathcal{N} = \{e_0, e_1, \ldots, e_k, e_{-1}, \ldots, e_{-k}\}$, with : $e_0 = \vec{0}$ and $\forall i, e_i \in \mathbb{Z}^d$,
then the configuration
where all cells are in state 0 except those that are a linear combination of the e_i is a **fixed point**.

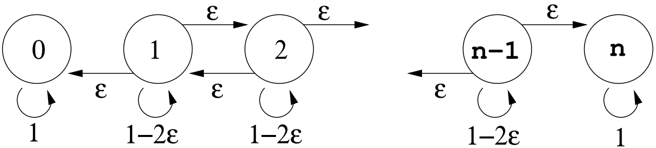
example :
$e_1 = (1, 0)$,
$e_2 = (1, 1)$,
$e_3 = (-1, 1)$.



Back

The shift : the two-zones case

					+								-		
0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0



Markov chain: number of cells in state 1, update prob. $\epsilon = 1/n$

"Step-forward" method: $T_0 = 0$, $T_n = 0$

$$T_i = \varepsilon \cdot (1 + T_{i+1}) + \varepsilon \cdot (1 + T_{i-1}) + (1 - 2\varepsilon) \cdot (1 + T_i)$$

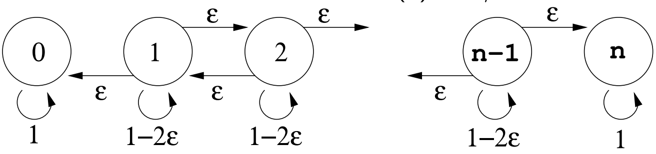
which solves as:

$$T_i = \frac{i(n-i)}{2\varepsilon}.$$

The shift case... with multiple zones

	+	-			+		-		+		-	+		-	
0	0	1	0	0	0	1	1	0	0	1	1	0	1	1	0

Same “Markov chain” but with $\varepsilon(t) \geq 1/n$!



Process is a **martingale**: $E[X_{t+1}|X_t] = X_t$ thus: $\forall t, E[X_t] = X_0$

Application: T time to reach a fixed point

$$E[X_T] = 0 \cdot \Pr[X_T = 0] + n . \Pr[X_T = n]$$

which leads to:

$$\Pr[X_T = n] = X_0/n$$

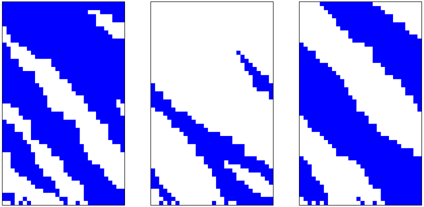
$\rightsquigarrow$ What about the convergence time ?

Central Lemma

For a stochastic process X_t, if:

- X_t is a martingale in $\{0, \cdots, k\}$,
- for $X_t \notin \{0, k\}$,
  proba to incr. or to decr. X_t by 1 is greater than ϵ,

then, average conv. time to $\{0, k\}$ upper-bounded by: $\frac{X_0(k-X_0)}{2\epsilon}$
when X_0 and k scale as n, $WECT = \Theta(n^2)$: **square scaling**



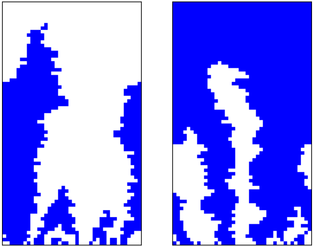
Evolutions of BDEF - 170

proof : examine $Y_t = {X_t}^2 - 2 \cdot \epsilon \cdot t$

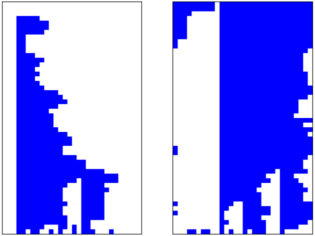
Back

Flip-if-not equal and traffic

Lemma applies, but different functions:
BCDEFG: $X_t = |x^t|_1 + Z(x^0) + D_t - E_t$



Evolutions of BCDEFG - 178



Evolutions of CDEG - 184

Quantitatively better, but not qualitatively : square scaling with n

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Fukś Density Classifier

For $p \in (0, 1/2]$, the local rule $\mathbf{C}_1$ has transition table:

(x, y, z)	000	001	010	011	100	101	110	111
$f(x, y, z)$	0	p	$1 - 2p$	$1 - p$	p	$2p$	$1 - p$	1

What is this rule doing ? hint : use the T-code...

Fukś Density Classifier

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What is this rule doing ? hint : use the T-code...
Interpretation: apply, for each cell independently:

- a left shift with probability p
- a right shift with probability p
- stay in the same state with probability $1-2p$

A convergence lemma

If

- $(x^t)_{t\in\mathbb{N}}$ is the evolution of stochastic CA with I.C. $x \in \mathcal{E}_n$
- M a mapping $M : \mathcal{E}_n \to \{0, \dots, m\}$
- (X_t) sequence of rand. var. $\forall t, X_t = M(x^t)$.
- the stochastic process (X_t) is a martingale on $\{0, \dots, m\}$, that is, for a filtration $\mathcal{F}_t$ adapted to (X_t), $\mathbb{E}\{\, \Delta X_{t+1} \,|\mathcal{F}_t \,\} = 0$,
- $X_t \in \{1, \dots, m-1\} \implies \mathrm{var}\{\, \Delta X_{t+1} \,\} > v$,

then:

$$\Pr\{X_T = m\} = \frac{q}{m}$$

and the absorbing time of the process $T(x) = \min\{t : X_t = 0 \text{ or } X_t = m\}$ is finite and obeys:

$$\mathbb{E}\{T(x)\} \leq \frac{q(m-q)}{v} \leq \frac{m^2}{4v}$$

where $q = \mathbb{E}\{X_0\} = M(x)$.

Application to Fukś rule

Lemma applies for $m = n$, $v = p$ and $M = |x|_1$.

=> proba. to attain 1^* is equal to the initial density.

Time to attain a fixed point ? For $n = 149$:

Application to Fukś rule

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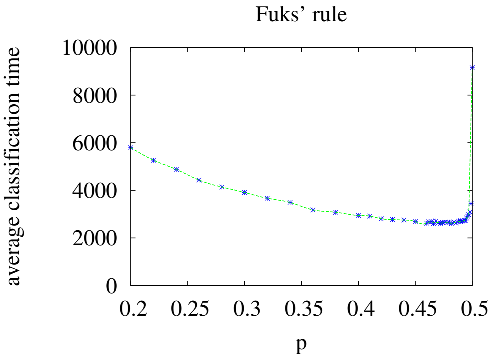
Time to attain a fixed point ? For $n = 149$:



surprise : increase for $p \to 1/2$! (checkerboards)
open problem : find the optimal value for p

Back

Schüle's rule

For $\varepsilon \in (0,1]$, Schüle's rule $\mathbf{C}_2$ has transitions:

(x,y,z)	000	001	010	011	100	101	110	111
$f(x,y,z)$	0	$1-\varepsilon$	$1-\varepsilon$	ε	$1-\varepsilon$	ε	ε	1

Interpret this rule

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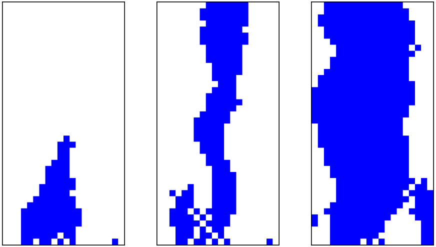
(x,y,z)	000	001	010	011	100	101	110	111
$f(x,y,z)$	0	$1-\varepsilon$	$1-\varepsilon$	ε	$1-\varepsilon$	ε	ε	1

Interpret this rule Apply, for each cell independently:

- a majority rule with probability ε
- a XOR (on 3 neighbours) with probability $1-\varepsilon$



3 evolutions of the rule $\mathbf{C}_2$ with $\varepsilon = 0.8$

Properties & Results

For $\varepsilon = 2/3$, lemma applies with $v = \varepsilon(1-\varepsilon)$ and $m = n$.

=> proba to go to **1** is still equal to the density and

$$T_{\mathrm{conv}} \leq \frac{n^2}{4\epsilon(1-\epsilon)} \leq 9/2 \cdot n^2$$

(authors used **mean-field** hypothesis for the density result)

For $\epsilon > 2/3$, the convergence to extremities is slightly improved: better than Fukś rule, but time still scales quadratically with n.

Back

A new rule for the density classification

For $\eta \in (0,1]$, consider the rule $\mathbf{C}_3$:

(x,y,z)	000	001	010	011	100	101	110	111
$f(x,y,z)$	0	0	0	1	$1-\eta$	1	η	1

Interpret this rule

A new rule for the density classification

For $\eta \in (0,1]$, consider the rule $\mathbf{C}_3$:

(x,y,z)	000	001	010	011	100	101	110	111
$f(x,y,z)$	0	0	0	1	$1-\eta$	1	η	1

Interpret this rule

Apply, for each cell independently:

- the **majority** rule with proba. $1-\eta$
- the **"traffic"** rule with proba. η

claim: system solves the DCP with an arbitrary precision

- stochastic system **but** some config. are perfectly classified
- for $\eta \to 0$, these configurations are attained with proba. that approaches 1

Archipelagos and the Traffic rule

1-*archipelago*: all 1s isolated

0	0	1	0	1	0	**0**	0	t=3
0	1	0	1	0	1	0	0	t=2
0	1	1	0	1	0	0	0	t=1
0	1	1	1	0	0	0	0	t=0

$\lfloor n/2 \rfloor$ steps, config. is archipelago !

Traffic rule / $\mathbf{C}_3$ rule

$\mathbf{C}_3$ and Traffic: successor of a q-archipelago is a q-archipelago
difference between $\mathbf{C}_3$ and Traffic is on some isolated 0s or 1s: they "vanish" with proba. ε

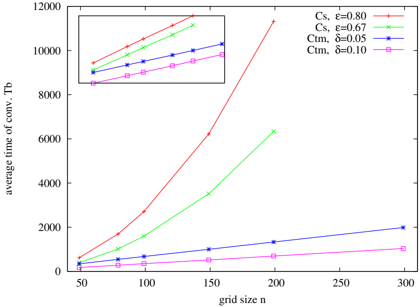
archipelagos are well-classified with probability 1 !

As $\eta \to 0$, we get closer to the Traffic rule, reach an archipelago.
Then, take all the time needed to go to the "right" fixed point.

And... empirical qualities and convergence times

Results for $n = 149$ and 10 000 samples.

model	setting	$Q_{\rm b}$ (in%)	$Q_{\rm d}$ (in%)	$T_{\rm b}$
$\mathbf{C}_1$	$p = 0.48$	53.3	75.0	**2652**
$\mathbf{C}_1$	$p = 0.5$	53.3	75.0	8985
$\mathbf{C}_2$	$\epsilon = 0.9$	56.6	85.8	11887
$\mathbf{C}_3$	$\eta = 0.1$	82.4	98.1	**517**
$\mathbf{C}_3$	$\eta = 0.01$	**91.0**	99.1	4950
$\mathbf{C}_3$	$\eta = 0.005$	93.4	99.3	9981



open problem: establish the **linear convergence time**

The synchronisation problem

objective :

from **any** initial condition, reach a "blinking uniform" configuration, i.e., $x \to \ldots \mathbf{0} \to \mathbf{1} \to \mathbf{0} \to \ldots$.

difficult ?

The synchronisation problem

objective :

from **any** initial condition, reach a "blinking uniform" configuration, i.e., $x \to \ldots \mathbf{0} \to \mathbf{1} \to \mathbf{0} \to \ldots.$

difficult ? 0 and 1 are not quiescent, the T-code contains A and H
any PCA with "positive rates" (other transitions) solves the problem

but... "efficient" solutions ?

idea: re-use the DCP solutions with a "**negative**" effect
$+$ plenty of other combinations

$\rightsquigarrow$ open problem: is there a rule with a **linear** convergence ?

The pattern Turing sym. breaking game (Buren game ?)

obj. : | from **any** initial condition, converge to $(01)^*$ or to $(10)^*$ | take even n

colonnes de Buren src: lejdd.fr

Show the "duality" of ECA `flip-if-not-equal` and ABCFGH - 23

other solutions : BCFG - 150 and BCGH - 30
phase transitions for α-asynch. updating

Openings...

stochastic CAs can be used for solving various problems...
many simple problems remain open (classification)
Computer science has been focused on rapidity

$\rightsquigarrow$ is there another path (robustness ?)

creativity =

taking advantage of the constructive role of randomness ?

beyond the deterministic sequential "machine"...

there is another Turing !